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# The Topp-Leone Burr-XII Distribution: Properties and Applications

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***Authors' contributions***

*This work was carried out in collaboration among both authors. Author HMR introduced the idea in a methodically structure, did the data analysis and drafted the manuscript. Author SAO assisted in building the study design and also did the final proofreading. Both authors managed the analyses of the study and literature searches and approved the final manuscript.*

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## Abstract

In this paper we introduce a new generalization of the Burr-XII distribution using the genesis of the Topp-Leone distribution and is named as Topp-Leone Burr-XII (TLBXII) distribution. The statistical properties of this distribution including the mean, variance, coefficient of variation, quantile function, median, ordinary and incomplete moments, skewness, kurtosis, moment and probability generating functions, reliability analysis, Lorenz, Bonferroni and Zenga curves, Rényi of entropy and order statistics are studied. We consider the method of maximum likelihood for estimating the model parameters and the observed information matrix is derived. Three real data sets are presented to demonstrate the effectiveness of the new model.

*Keywords: Burr-XII distribution; maximum likelihood estimation; order statistics.*

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# 1 Introduction

The Burr-XII distribution was first emerged by Burr [1]. The researchers have more attention to the Burr-XII distribution due to its wide applications in various area such as; failure time modelling, finance, acceptance sampling plan, reliability and hydrology.

Different extentions of this distribution have been derived for examples; Paranaíba et al. [2] discussed the beta Burr XII distribution. The Kumaraswamy Burr XII distribution presented by Paranaíba et al. [3]. Antonio et al. [4] introduced the McDonald Burr XII distribution. Mead [5] proposed the beta exponentiated Burr XII distribution.

The random variable $X$ with Burr-XII distribution has the distribution function (cdf) given by

$$G(x;\alpha,\beta)=1-(1+x^{\beta})^{-\alpha}, \qquad x>0,\ \alpha,\beta>0 \tag{1}$$

where $\beta$ and $\alpha$ are two shape parameters. The probability density function (pdf) corresponding to Eq. (1) takes the form

$$g(x;\alpha,\beta)=\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\alpha+1)}, \qquad x>0,\ \alpha,\beta>0 \tag{2}$$

The main object of this paper is to propose a new model, so called, Topp-Leone Burr-XII (TLBXII) distribution and studied some of its various statistical properties. In addition, the parameters of the new distribution are estimated by using the maximum likelihood estimation. Three real data sets are applied to show the usefullness of the new distribution.

The layout of this paper is as follow. In Section 2, we define the TLBXII distribution and derive some associated reliability functions . The limit of the TLBXII distribution is studied in Section 3. The expansion of TLBXII distribution is discussed in Section 4. In Section 5, some statistical properties of the new model are investigated. In Section 6, the maximum likelihood estimates for the model parameters are obtained and the observed information matrix is derived. In Section 7, three applications of the new distribution are applied. Some concluding remarks have been given in the last Section.

# 2 The Topp-Leone Burr-XIIDistribution

In this section, we introduce the TLBXII distribution and its sub-models. Some reliability functions corresponding to the TLBXII distribution are also discussed. Consider the Topp-Leone generated family of distributions proposed by Al-Shomrani et al. [6] with its cumulative distribution function (cdf) and probability density function (pdf) given by,

$$F_{TL-G}(x)=\left[G(x)\right]^{\lambda}\left[2-G(x)\right]^{\lambda}, \qquad \lambda>0 \tag{3}$$

$$f_{TL-G}(x)=2\lambda g(x)\bar{G}(x)\left[G(x)\right]^{\lambda-1}\left[2-G(x)\right]^{\lambda-1}, \qquad \lambda>0 \tag{4}$$

where $G(x)$ is the baseline distribution function, $\bar{G}(x)=1-G(x)$ and $g(x)=\partial G(x)/\partial x$ is the baseline density function. Inserting Eq. (1) in Eq. (3), we obtain a new distribution, so-called the Topp-Leone Burr-XII (TLBXII) distribution with cdf given from

$$F(x;\alpha,\beta,\lambda)=\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda} \tag{5}$$

The pdf corresponding to Eq. (5) is given by

$$f(x;\alpha,\beta,\lambda)=2\lambda\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(2\alpha+1)}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda-1} \tag{6}$$

for $x>0, \alpha>0, \beta>0$ and $\lambda>0$.

For the reliability analysis, the reliability function $R(x)$, hazard function $h(x)$, inverse hazard function $h_r(x)$ and cumulative hazard function $H(x)$ for the TLBXII distribution are given from

$$R(x)=1-F(x)=1-\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda}, \tag{7}$$

$$h(x)=\frac{f(x)}{R(x)}=\frac{2\lambda\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(2\alpha+1)}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda-1}}{1-\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda}}, \tag{8}$$

$$h_r(x)=\frac{f(x)}{F(x)}=\frac{2\lambda\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(2\alpha+1)}}{1-\left(1+x^{\beta}\right)^{-2\alpha}} \tag{9}$$

and

$$H(x)=-\ln R(x)=-\ln\left\{1-\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda}\right\} \tag{10}$$

### 2.1 Sub-models

The following distributions can be obtained as special cases of the TLBXII distribution:

1. If $\lambda=1$, Eq. (6) reduces to the two-parameter Burr-XII distribution.
2. When $\lambda=\alpha=1$, Eq. (6) represents one parameter Burr-XII distribution.
3. Suppose $\lambda=\beta=1$, then we obtain the Lomax distribution.
4. Setting $\lambda=\alpha=\beta=1$, the TLBXII distribution is reduced to the log-logistic distribution.

## 3 The Limit of the Topp-Leone Burr-XII Distribution

The limit of the TLBXII distribution when $x\to 0$ is 0 and when $x\to\infty$ is 0. We can show this by taking the limit of Eq. (6) as follows:

$$\lim_{x\to 0} f(x)=\left(\lim_{x\to 0} 2\lambda\alpha\beta\right)\left(\lim_{x\to 0} x^{\beta-1}\right)\left[\lim_{x\to 0}\left(1+x^{\beta}\right)^{-(2\alpha+1)}\right]\left\{\lim_{x\to 0}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda-1}\right\}=0$$

Because $\lim_{x\to 0} x^{\beta-1} = 0$ and $\lim_{x\to 0}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda-1} = 0$.

Likewise, as $x \to \infty$, we can observe that by replacing the limit $x \to 0$ with $x \to \infty$, we get

$$\lim_{x\to\infty} f(x) = 0$$

Because $\lim_{x\to\infty}\left(1+x^{\beta}\right)^{-(2\alpha+1)} = 0$.

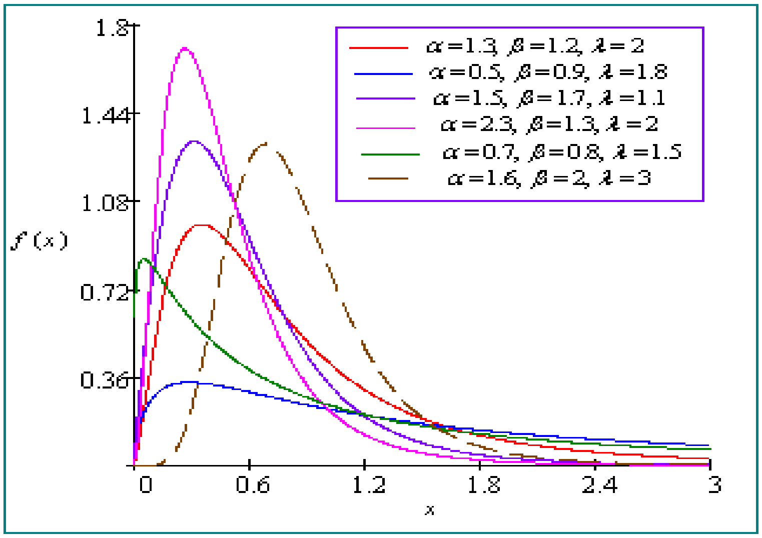


**Fig. 1. The pdf of the TLBXII distribution for different values of the parameters**

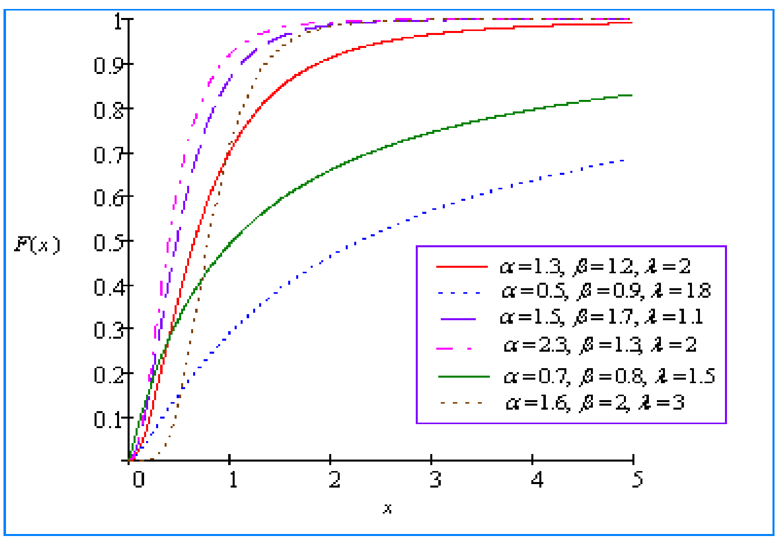


**Fig. 2. The cdf of the TLBXII distribution for different values of the parameters**

# 4 Expansions for the Topp-Leone Burr-XII Distribution

We can expand the cdf and pdf corresponding to the TLBXII distribution in terms of an infinite (or finite) weighted sums of cdf 's and pdf 's of random variables having Burr-XII distributions respectively. For $\lambda$ is a real non-integer,then we have the series representation

$$(1-m)^{\lambda-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(\lambda)}{j\,!\Gamma(\lambda-j)}m^j \tag{11}$$

Therefore, the cdf of TLBXII distribution can be expressed as follows:

$$F(x;\alpha,\beta,\lambda)=\sum_{j=0}^{\infty}z_j\,R(x;2\alpha j,\beta) \tag{12}$$

Where

$$z_j=\frac{(-1)^j\,\Gamma(\lambda+1)}{j\,!\Gamma(\lambda-j)}$$

and $R(x;2\alpha j,\beta)$ deotes the reliability function of Burr-XII distribution with parameters $2\alpha j$ and $\beta$.

Similarly, we can express the pdf in Eq. (6) as below

$$f(x;\alpha,\beta,\lambda)=\sum_{j=0}^{\infty}w_j\,H\left(x;2\alpha(j+1),\beta\right) \tag{13}$$

Where

$$w_j=\frac{(-1)^j\,\Gamma(\lambda+1)}{\Gamma(j+2)\Gamma(\lambda-j)}$$

and $H\left(x;2\alpha(j+1),\beta\right)$ deotes the cdf of Burr-XII distribution with parameters $2\alpha(j+1)$ and $\beta$. If $\lambda$ is an integer, then the summations in Eqs. (12) and (13) are stoped at $\lambda-1$.

# 5 Statistical Properties

In this section, we discuss some statistical properties of the proposed distribution such as; mean, variance, coefficient of variation, quantile function, median, ordinary and incomplete moments, skewness, kurtosis, moment and probability generating functions, Lorenz, Bonferroni and Zenga curves, Rényi of entropy and order statistics.

## 5.1 Ordinary moments

Suppose $X$ is a random variable distributed according to TLBXII distribtion, then the ordinary moments, say $\mu_r'$ is given by

$$\mu_r' = E(X^r) = 2\lambda\alpha\beta\int_0^\infty x^{r+\beta-1}\left(1+x^{\beta}\right)^{-(2\alpha+1)}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda-1}dx$$

By using the binomial expansion in the last term of above integrand, we get

$$\mu_r' = 2\lambda\alpha\beta\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\int_0^\infty x^{r+\beta-1}\left(1+x^{\beta}\right)^{-[2\alpha(j+1)+1]}dx$$

Let $z = \left(1+x^{\beta}\right)^{-1}$ in the above equation, so we have

$$\mu_r' = 2\lambda\alpha\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\int_0^1 z^{2\alpha(j+1)-(r/\beta)-1}\left(1-z\right)^{r/\beta}dz$$

$$= 2\lambda\alpha\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\,\beta\left[2\alpha(j+1)-r/\beta,(r/\beta)+1\right] \tag{14}$$

where $\beta(.,.)$ is the beta function. Substituting $r = 1,2$ in Eq. (14), then we get the mean and variance respectively as follows:

$$\mu_1' = 2\lambda\alpha\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\,\beta\left[2\alpha(j+1)-1/\beta,(1/\beta)+1\right] \tag{15}$$

and

$$v(x) = 2\lambda\alpha\left\{\begin{array}{l}\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\,\beta\left[2\alpha(j+1)-2/\beta,(2/\beta)+1\right]\\ -2\lambda\alpha\left[\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\,\beta\left[2\alpha(j+1)-1/\beta,(1/\beta)+1\right]\right]^2\end{array}\right\} \tag{16}$$

## 5.2 Coefficients of variation, skewness and kurtosis

The coefficients of variation, skewness and kurtosis of the TLBXII distribution are given respectively as follows:

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\begin{array}{l}\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\,\beta\left[2\alpha(j+1)-2/\beta,(2/\beta)+1\right]\\ -2\lambda\alpha\left[\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\,\beta\left[2\alpha(j+1)-1/\beta,(1/\beta)+1\right]\right]^2\end{array}}}{\left(2\lambda\alpha\right)^{1/2}\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\,\beta\left[2\alpha(j+1)-1/\beta,(1/\beta)+1\right]}, \tag{17}$$

$$\varpi_1 = \frac{\mu_3'}{(\mu_2')^{3/2}} = \frac{\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j \beta\left[2\alpha(j+1)-3/\beta,(3/\beta)+1\right]}{(2\lambda\alpha)^{1/2}\left\{\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j \beta\left[2\alpha(j+1)-1/\beta,(1/\beta)+1\right]\right\}^{3/2}} \tag{18}$$

And

$$\varpi_2 = \frac{\mu_4'}{(\mu_2')^{2}} = \frac{\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j \beta\left[2\alpha(j+1)-4/\beta,(4/\beta)+1\right]}{2\lambda\alpha\left\{\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j \beta\left[2\alpha(j+1)-1/\beta,(1/\beta)+1\right]\right\}^{2}} \tag{19}$$

## 5.3 Quantile function

The quantile function of the TLBXII distribution, say $Q(\mu) = F^{-1}(\mu)$ random variable can be obtained by inverting Eq. (5) as

$$Q(\mu) = \left\{\left[\mu^{1/\lambda} - 1\right]^{-1/2\alpha} - 1\right\}^{1/\beta} \tag{20}$$

Simulating the TLBXII random variable is straightforward. If $U$ is a uniform variate on the interval (0,1), then the random variable $X = Q(U)$ follows Eq. (6). From Eq. (20), we can deduct that the median $m$ of $X$ is $m = Q(1/2)$.

## 5.4 Incomplete moments

Suppose $X$ is a random variable having the TLBXII distribution, then the $r^{th}$ incomplete moments denoted as $m_r(z)$ can be obtained as follows:

$$m_r(z) = \int_0^z x^r f(x)dx$$
$$= 2\lambda\alpha\beta\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j \int_0^z x^{r+\beta-1}\left(1+x^{\beta}\right)^{-[2\alpha(j+1)+1]}dx$$

Based on the binomial expansion to the last factor, we get

$$m_r(z) = 2\lambda\alpha\sum_{j=0}^{\infty}\sum_{i=0}^{r/\beta}\binom{\lambda-1}{j}\binom{r/\beta}{i}(-1)^{j+i}\int_{\left(1+z^{\beta}\right)^{-1}}^{1} y^{2\alpha(j+1)+i-\frac{r}{\beta}-1}dy$$

$$= 2\lambda\alpha\sum_{j=0}^{\infty}\sum_{i=0}^{r/\beta}\binom{\lambda-1}{j}\binom{r/\beta}{i}(-1)^{j+i}\left[1-\left(1+z^{\beta}\right)^{r/\beta-[2\alpha(j+1)+i]}\right] \tag{21}$$

## 5.5 Moment and probability generating functions

The moment generating function, say $M_x(t)$ of the TLBXII distribution can be obtained as follows:

$$M_x(t) = E(e^{tx}) = 2\lambda\alpha\beta\int_0^\infty e^{tx}x^{\beta-1}\left(1+x^\beta\right)^{-(2\alpha+1)}\left[1-\left(1+x^\beta\right)^{-2\alpha}\right]^{\lambda-1}dx$$

$$= 2\lambda\alpha\beta\sum_{j=0}^\infty\binom{\lambda-1}{j}(-1)^j\int_0^\infty e^{tx}x^{\beta-1}\left(1+x^\beta\right)^{-[2\alpha(j+1)+1]}dx$$

Using $e^{tx} = \sum_{h=0}^\infty \frac{t^h x^h}{h!}$, then we obtain

$$M_x(t) = 2\lambda\alpha\sum_{j,h=0}^\infty\binom{\lambda-1}{j}\frac{(-1)^j t^h}{h!}\beta\left[2\alpha(j+1)-h/\beta,(h/\beta)+1\right] \quad (22)$$

Similarly, the probability generating function denoted as $M_{[x]}(t)$ of the TLBXII distribution can be derived as below:

$$M_{[x]}(t) = E(t^x) = 2\lambda\alpha\beta\int_0^\infty t^x x^{\beta-1}\left(1+x^\beta\right)^{-(2\alpha+1)}\left[1-\left(1+x^\beta\right)^{-2\alpha}\right]^{\lambda-1}dx$$

$$= 2\lambda\alpha\beta\sum_{j=0}^\infty\binom{\lambda-1}{j}(-1)^j\int_0^\infty t^x x^{\beta-1}\left(1+x^\beta\right)^{-[2\alpha(j+1)+1]}dx$$

Using $t^x = \sum_{\ell=0}^\infty \frac{(\ln t)^\ell x^\ell}{\ell!}$, then we have

$$M_{[x]}(t) = 2\lambda\alpha\sum_{j,\ell=0}^\infty\binom{\lambda-1}{j}\frac{(-1)^j(\ln t)^\ell}{\ell!}\beta\left[2\alpha(j+1)-\ell/\beta,(\ell/\beta)+1\right] \quad (23)$$

## 5.6 Lorenz, Bonferroni and Zenga curves

The Lorenz, Bonferroni and Zenga curves have different applications in many fields such as insurance, medicine, demography and economics (see Kleiber & Kotz [7]). Oluyede & Rajasooriya [8] defined the Lorenz $L_F(x)$, Bonferroni $B\left(F(x)\right)$ and Zenga $A(x)$ curves respectively as follows:

$$L_F(x) = \frac{1}{E(x)}\int_0^x t f(t)dt, B\left(F(x)\right) = \frac{1}{F(x)E(x)}\int_0^x t f(t)dt = \frac{L_F(x)}{F(x)}, A(x) = 1-\left[\frac{M^-(x)}{M^+(x)}\right]$$

Where

$$M^-(x) = \frac{1}{F(x)}\int_0^x t f(t)dt \text{ and } M^+(x) = \frac{1}{1-F(x)}\int_x^\infty t f(t)dt$$

Therefore, these quantities for the TLBXII distribution are obtained below

$$L_F(x) = \frac{\Delta_1}{\Delta_2}, \tag{24}$$

$$B(F(x)) = \frac{\Delta_1}{\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda} \Delta_2} \tag{25}$$

And

$$A(x) = 1 - \frac{\Delta_1 \Delta_3}{\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda} \Delta_4} \tag{26}$$

Where

$$\Delta_1 = \sum_{j=0}^{\infty} \sum_{i=0}^{1/\beta} \frac{\binom{\lambda-1}{j}\binom{1/\beta}{i}(-1)^{j+i}}{2\alpha(j+1)+i-(1/\beta)} \left[1-\left(1+x^{\beta}\right)^{1/\beta-[2\alpha(j+1)+i]}\right],$$

$$\Delta_2 = \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \, \beta\left[2\alpha(j+1)-1/\beta, (1/\beta)+1\right],$$

$$\Delta_3 = 1 - \left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda}$$

$$\Delta_4 = \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \, \beta_{\left[1+x^{\beta}\right]^{-1}} \left[2\alpha(j+1)-1/\beta, (1/\beta)+1\right]$$

Where $\beta_y(a,b) = \int_0^y u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function.

## 5.7 Rényi entropy

The entropy represents to the amount of uncertainty contained in a random variable. The Rényi entropy has broad applications in different area such as; statistics, computer science and econometrics. The Rényi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta}\left[\log I(\delta)\right],$$

where $I(\delta) = \int f^{\delta}(x)dx$, $\delta > 0$ and $\delta \neq 0$. Using Eq. (6) yields.

$$I(\delta)=2^{\delta}\lambda^{\delta}\alpha^{\delta}\beta^{\delta}\int_{0}^{\infty}x^{\delta(\beta-1)}\left(1+x^{\beta}\right)^{-\delta(2\alpha+1)}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\delta(\lambda-1)}dx$$

$$=2^{\delta}\lambda^{\delta}\alpha^{\delta}\beta^{\delta}\sum_{j=0}^{\infty}\binom{\delta(\lambda-1)}{j}(-1)^{j}\int_{0}^{\infty}x^{\delta(\beta-1)}\left(1+x^{\beta}\right)^{-[2\alpha(\delta+j)+\delta]}dx$$

$$=2^{\delta}\lambda^{\delta}\alpha^{\delta}\beta^{\delta-1}\sum_{j=0}^{\infty}\binom{\delta(\lambda-1)}{j}(-1)^{j}\beta\left[2\alpha(\delta+j)+(\delta-1)/\beta,\left(\delta(\beta-1)+1\right)/\beta,\ \right]$$

Therefore, the Rényi entropy is given below

$$I_{R}(\delta)=\frac{1}{1-\delta}\left\{\begin{array}{l}\delta\left[\log(2)+\log(\lambda)+\log(\alpha)\right]+(\delta-1)\log(\beta)\\ +\sum_{j=1}^{\infty}\log\binom{\delta(\lambda-1)}{j}(-1)^{j}\beta\left[2\alpha(\delta+j)+(\delta-1)/\beta,\left(\delta(\beta-1)+1\right)/\beta,\ \right]\end{array}\right\} \qquad (27)$$

## 5.8 Order statistics

Order statistics play an important role in probability and statistics. Let $x_{1,n}\le x_{2:n},\ldots\le x_{n:n}$ be the ordered sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{k:n}$, the $k^{th}$ order statistics is given by

$$f_{X_{k:n}}(x)=\frac{n!}{(k-1)!(n-k)!}f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k},\ r=1,2,\ldots,n$$

Then, the pdf the $k^{th}$ order TLBXII random variable $X_{k:n}$ can be obtained by using Eqs.(5) and (6) in above equation to be

$$f_{X_{k:n}}(x)=\frac{2n!\lambda\alpha\beta}{(k-1)!(n-k)!}x^{\beta-1}\left(1+x^{\beta}\right)^{-(2\alpha+1)}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda(k+j-1)} \qquad (28)$$

Therefore, the pdf the $1^{th}$ order TLBXII random variable $X_{1:n}$ is given by

$$f_{X_{1:n}}(x)=2n\lambda\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(2\alpha+1)}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{\lambda(j+1)-1} \qquad (29)$$

Also, the pdf the $n^{th}$ order TLBXII random variable $X_{n:n}$ is given from

$$f_{X_{n:n}}(x)=2n\lambda\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(2\alpha+1)}\left[1-\left(1+x^{\beta}\right)^{-2\alpha}\right]^{n\lambda-1} \qquad (30)$$

Moreover, the joint distribution of two order statistics $X_{k:n}\le X_{s:n}$ is given by

$$f_{X_{k:s:n}}(x_1,x_2)=\frac{n!}{(k-1)!(s-k-1)!(n-s)!}f(x_1)f(x_2)\left[F(x_1)\right]^{k-1}\left[F(x_2)-F(x_1)\right]^{s-k-1}\left[1-F(x_2)\right]^{n-s}$$

Then, for the TLBXII distribution we obtain

$$f_{X_{k:s:n}}(x_1,x_2)=\frac{4n!\lambda^2\alpha^2\beta^2}{(k-1)!(s-k-1)!(n-s)!}\,x_1^{\beta-1}x_2^{\beta-1}\left(1+x_1^{\beta}\right)^{-(2\alpha+1)}\left(1+x_2^{\beta}\right)^{-(2\alpha+1)}$$

$$\times\sum_{i=0}^{n-s}\sum_{j=0}^{s-k-1}\binom{n-s}{i}\binom{s-k-1}{j}\left[1-\left(1+x_1^{\beta}\right)^{-2\alpha}\right]^{\lambda(k+j)-1}\left[1-\left(1+x_2^{\beta}\right)^{-2\alpha}\right]^{\lambda(s-j-k)-1} \tag{31}$$

# 6 Estimation of Parameters

In this section, we obtain the maximum likelihood estimates (MLEs) and the observed information matrix of the TLBXII distribution. Let $x_1,x_2,...,x_n$ be an independent random sample from the TLBXII distribution, then the corresponding log-likelihood function is given by

$$\ell=n\left[\ln(2)+\ln(\lambda)+\ln(\alpha)+\ln(\beta)\right]+(\beta-1)\sum_{i=1}^{n}\ln(x_i)-(2\alpha+1)\sum_{i=1}^{n}\ln(A_i)+(\lambda-1)\sum_{i=1}^{n}\ln(1-A_i^{-2\alpha}) \tag{32}$$

where $A_i=1+x_i^{\beta}$.

The components of the score vector $\nabla\ell=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\lambda}\right)$ are given below:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-2\sum_{i=1}^{n}\ln(A_i)+2(\lambda-1)\sum_{i=1}^{n}\left[\frac{A_i^{-2\alpha}\ln(A_i)}{1-A_i^{-2\alpha}}\right], \tag{33}$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln(x_i)-(2\alpha+1)\sum_{i=1}^{n}\left[\frac{x_i^{\beta}\ln(x_i)}{A_i}\right]+2\alpha(\lambda-1)\sum_{i=1}^{n}\left[\frac{x_i^{\beta}A_i^{-(2\alpha+1)}\ln(x_i)}{1-A_i^{-2\alpha}}\right] \tag{34}$$

and

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left[1-A_i^{-2\alpha}\right] \tag{35}$$

MLEs, say $(\hat{\alpha},\hat{\beta},\hat{\lambda})$ of $(\alpha,\beta,\lambda)$ can be obtained by solving the system of nonlinear Eqs (33) through (35). These equations cannot be solved analytically and it needed numeric techniques such as NR (Newton-Raphson) and BFGS (Broyden-Fletcher-Goldfarb-Shanno).

For interval estimation and testing of hypotheses of the model parameters $(\alpha,\beta,\lambda)$, we derive the $3\times3$ observed information matrix $J(\Theta)=\{J_{mn}\}$ (for $m,n=\alpha,\beta,\lambda$) to be

$$J(\Theta)=\begin{bmatrix}J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\lambda}\\ J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\lambda}\\ J_{\lambda\alpha} & J_{\lambda\beta} & J_{\lambda\lambda}\end{bmatrix}$$

whose elements are obtained in Appendix A.

# 7 Applications

In this section, we introduce three applications of the TLBXII distribution to three real data sets. The first data set from Murthy et al. [9]. This data set consists of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h. The data are: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981,2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890,4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255,1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615,2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300,3.344, 4.602, 1.757, 2.324, 3.376, 4.663. This data is previously analyzed by Ramos et al. [10] and Tahir et al. [11].

The second data set consists of 63 observations of the strengths of 1.5 cm glass fibers which obtained by workers at the UK National Physical Laboratory. The data are: 0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. This data have also been used by Smith & Naylor [12] and Merovci et al. [13].

The third data set consists of 63 observations of the gauge lengths of 10 mm from Kundu & Raqab [14]. The data set are: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397,2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659,2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139,3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435,3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225,4.395, 5.020. This data was also used by Afify et al. [15].

We use these two data sets to compare the fit of the new model, TLBXII with three models: Burr-XII, Lomax and log-logistic. First, we obtain the maximum likelihood estimates (MLEs) for the unknown parameters of each model and then comparing the results via goodness-of-fit statistics AIC (Akaike information criterion), AICC (corrected Akaike information criterion), CAIC (consistent Akaike information criterion) and BIC (Bayesian information criterion). The better model corresponds to smaller AIC, AICC, CAIC and BIC values.

Where

$$AIC = 2k - 2\hat{\ell}(.), \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

And

$$CAIC = \frac{2kn}{n-k-1} - 2\hat{\ell}(.), \qquad BIC = k\log(n) - 2\hat{\ell}(.)$$

where $\hat{\ell}(.)$ denotes the log-likelihood function evaluated at the MLEs, $k$ is the number of parameters, and $n$ is the sample size. The MLEs and the values of AIC, AICC, CAIC and BIC displayed in Tables (1-2).

**Table 1. MLEs for TLBXII, Burr-XII, Lomax and log-logistic models and the statistics AIC, AICC, CAIC and BIC; first data set**

| Model | Estimates | | | Statistics | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $-2\hat{\ell}$ | *AIC* | *AICC* | *CAIC* | *BIC* |
| TLBXII | 0.799 | 1.257 | 4.999 | **319.934** | **325.934** | **326.234** | **326.234** | **333.227** |
| Burr-XII | 0.352 | 3.171 | ------- | 338.854 | 342.854 | 343.003 | 343.003 | 347.716 |
| Lomax | 2002 | 0.0002 | ------- | 325.788 | 329.788 | 329.936 | 329.936 | 334.65 |
| log-logistic | 1.299 | 2.000 | ------- | 446.313 | 450.313 | 450.462 | 455.462 | 455.175 |

**Table 2. MLEs for TLBXII, Burr-XII, Lomax and log-logistic models and the statistics AIC, AICC, CAIC and BIC; second data set**

| Model | Estimates | | | Statistics | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $-2\hat{\ell}$ | *AIC* | *AICC* | *CAIC* | *BIC* |
| TLBXII | 1.295 | 2.073 | 13.125 | **71.764** | **77.764** | **78.171** | **78.171** | **84.194** |
| Burr-XII | 0.321 | 7.482 | ------- | 97.442 | 101.442 | 101.642 | 101.642 | 105.729 |
| Lomax | 515.573 | 0.0013 | ------- | 177.777 | 181.777 | 181.977 | 181.977 | 186.063 |
| log-logistic | 1.018 | 3.570 | ------- | 138.647 | 142.647 | 142.847 | 142.847 | 146.934 |

**Table 3. MLEs for TLBXII, Burr-XII, Lomax and log-logistic models and the statistics AIC, AICC, CAIC and BIC; third data set**

| Model | Estimates | | | Statistics | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $-2\hat{\ell}$ | *AIC* | *AICC* | *CAIC* | *BIC* |
| TLBXII | 0.350 | 7.032 | 138.705 | **118.923** | **124.923** | **125.330** | **125.330** | **131.352** |
| Burr-XII | 0.063 | 14.390 | ------- | 276.540 | 280.254 | 280.454 | 280.454 | 284.540 |
| Lomax | 609.483 | 0.0005 | ------- | 266.991 | 270.991 | 271.191 | 271.191 | 275.277 |
| log-logistic | 1.402 | 2.000 | ------- | 430.562 | 434.562 | 434.762 | 434.762 | 438.848 |

From Tables 1-3, it has been observed that the TLBXII model has the smallest values for the AIC, AICC, CAIC and BIC statistics among all fitted distributions. Consequently, we can conclude that the TLBXII distribution provides a significantly better fit than the other models. All computations were performed using the MATH-CAD PROGRAM.

## 8 Conclusion

This paper introduces a new distribution namely the Topp-Leone Burr-XII (TLBXII) distribution which is considered a new extension of the Burr-XII distribution. Different properties of the new distribution are studied including the mean, variance, coefficient of variation, quantile function, median, ordinary and incomplete moments, skewness, kurtosis, moment and probability generating functions, reliability, hazard, reverse hazard and cumulative hazard functions, Lorenz, Bonferroni and Zenga curves, Rényiof entropy and order statistics. The parameters of the new distribution are estimated by using the maximum likelihood approach and the observed Fisher information matrix is obtained. Three real data sets are applied to demonstrate that the new distribution can provide a better fit than other known distributions.

## Competing Interests

Authors have declared that no competing interests exist.

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# Appendix A

The elements of the observed information matrix are given below

$$J_{\alpha\alpha} = \frac{-n}{\alpha^2} - 4(\lambda - 1)\sum_{i=1}^{n}\left[\frac{\left(\ln(A_i)\right)^2}{A_i^{2\alpha}\left(1 - A_i^{-2\alpha}\right)^2}\right],$$

$$J_{\alpha\beta} = -2\sum_{i=1}^{n}\left[\frac{x_i^{\beta}\ln(x_i)}{A_i}\right] + 2(\lambda - 1)\sum_{i=1}^{n}\left\{\frac{x_i^{\beta}\ln(x_i)\left[\left(A_i^{2\alpha} - 1\right)\left(1 - 2\alpha\ln(A_i)\right) - 2\alpha\ln(A_i)\right]}{A_i^{4\alpha+1}\left[1 - A_i^{-2\alpha}\right]^2}\right\},$$

$$J_{\alpha\lambda} = 2\sum_{i=1}^{n}\left[\frac{\ln(A_i)}{A_i^{2\alpha} - 1}\right],$$

$$J_{\beta\beta} = \frac{-n}{\beta^2} - (2\alpha + 1)\sum_{i=1}^{n}\left[\frac{x_i^{\beta}\left(\ln(x_i)\right)^2}{A_i^2}\right] + 2\alpha(\lambda - 1)\sum_{i=1}^{n}\left\{\frac{x_i^{\beta}\left(\ln(x_i)\right)^2\left[A_i^{2\alpha}\left(1 - 2\alpha x_i^{\beta}\right) - 1\right]}{A_i^{2(\alpha+2)}\left[1 - A_i^{-2\alpha}\right]^2}\right\}$$

$$J_{\lambda\lambda} = \frac{-n}{\lambda^2}$$



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