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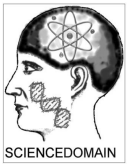

Global Stability of Equilibrium Points of Typhoid Fever Model with Protection

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

A non-linear mathematical model of typhoid fever diseases incorporating protection is hereby considered to study the global stability of equilibrium points. To study the global stability of the disease free equilibrium point and endemic equilibrium point, the method by Castillo-Chavez and a suitable Lyapunov function are used respectively. The disease free equilibrium point was found not to be globally asymptotically stable while the endemic equilibrium point is globally asymptotically stable. This implies that the disease transmission can be kept quiet low or manageable with minimal deaths in the presence of protection.

Keywords: Reproduction number; protection; typhoid fever; equilibrium point; lyapunov function; global stability.

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1 Introduction

Typhoid fever is a bacterial infection that is transmitted through food and water contaminated with faeces and urine of an infected patient or a carrier [1]. Signs and symptoms includes; sustained fever, poor appetite, vomiting , severe headache and fatigue. Incubation period for typhoid fever is about 7-14 days. Typhoid fever is an underestimated global health problem whose impact is difficult to estimate because the clinical picture is confused with other infections. An estimated 17 million cases of typhoid were reported worldwide resulting in 0.6 million deaths annually [2]. A mathematical model of typhoid fever disease incorporating protection against infection is presented by [3]. The local stability of the disease free equilibrium (DFE) and disease endemic equilibrium (DEE) points were analysed and were found to be locally asymptotically stable.

The stability of an equilibrium point determines whether or not, the solutions nearby the equilibrium point remains nearby, gets closer or get further away. Thus, an equilibrium point can be categorized as either stable or unstable or a saddle point [4].

The asymptotic stability regions of the equilibrium point of models involving temporary immunity, disease related fatalities, carriers, migration,dissimilar interacting groups and transmission by vectors were determined [5, 6]. Lyapunov method is successfully used to prove the global stability of endemic equilibrium. This method consist of finding one function called the Lyapunov function that is positive definite and its derivative along the trajectories is negative. Different Lyapunov functions[7, 8] are constructed for SIR, SIS and SIRS using combinations of suitable composite quadratic, simple quadratic and logarithmic function and presented. Global stability conditions for DFE when $R_0 < 1$ are presented by [9]. According to [10], The disease free equilibrium is globally asymptotically stable when the reproduction number is less than unity and the endemic equilibrium is globally asymptotically stable when the reproduction number is greater than unity.

In this work, the global stability of the DFE and DEE of the model in [3] are analyzed.

2 The Model Preliminaries and Analysis

The model herewith considered was formulated in [3] and is defined as

$$\begin{aligned} \frac{dP}{dt} &= \alpha\Lambda - (\gamma+\mu)P \\ \frac{dS}{dt} &= (1-\alpha)\Lambda + \gamma P - (\lambda+\mu)S \\ \frac{dI}{dt} &= \lambda S - (\delta+\beta+\mu)I \\ \frac{dT}{dt} &= \beta I - \mu T \end{aligned} \tag{2.1}$$

$$N(t) = P(t) + S(t) + I(t) + T(t) \tag{2.2}$$

where S is the class of susceptible individuals, P is the class of protected individuals , I is the class of infected individuals and T the class of treated individuals. $\alpha\Lambda$ is the recruitment rate into the class of individuals protected against typhoid, $(1-\alpha)\Lambda$ is the recruitment rate into the class of individuals susceptible to typhoid, μ is the natural mortality rate, δ is the disease induced mortality rate, β is the treatment rate. All population compartments are non negative $\forall t > 0$ in the feasible region Γ where $S(t), P(t), I(t), T(t) \in \Gamma \subset \mathbb{R}^4_+$ Since we are dealing with a population. It can be shown that all the solutions are bounded in Γ, $\forall t > 0$ such that $0 \leq N \leq \frac{\Lambda}{\mu}$. Thus the model is

epidemiologically well posed in the region Γ and can be analysed [3]. The force of infection is given by

$$\lambda = \frac{\pi\theta(1-\vartheta)I}{N}. \tag{2.3}$$

where π be defined as the probability rate of acquiring typhoid fever disease, θ is the contact rate of infection and ϑ is the probability of success of protection against typhoid fever disease and the effective reproduction number is

$$R_0 = \frac{\pi\theta(1-\vartheta)}{\mu+\delta+\beta} \tag{2.4}$$

2.1 Disease free equilibrium point and endemic equilibrium point

At the DFE we let $P=0$, $I=0$ and $T=0$, solving Equation 2.1 for S we obtain

$$S = \frac{\Lambda}{\mu}. \tag{2.5}$$

The disease-free equilibrium point E^0 is given by

$$E^0 = (0, \frac{\Lambda}{\mu}, 0, 0). \tag{2.6}$$

To calculate the endemic equilibrium point, we set P, S, I, T not equal to zero.

$$\begin{aligned}
P^* &= \frac{\alpha\Lambda}{\mu+\gamma} \\
S^* &= \frac{(\frac{\pi(-1+\vartheta)\theta(\alpha\Lambda\mu+I(\beta+\mu)(\gamma+\mu)}{\mu(\gamma+\mu)(\beta+\delta+\pi(-1+\vartheta)\theta+\mu)})(\mu+\delta+\beta)}{\pi\theta(1-\vartheta)} \\
I^* &= \frac{\Lambda(\gamma(\beta+\delta+\pi(-1+\vartheta)\theta)+\beta+\gamma+\delta-\pi(-1+\alpha)(-1+\vartheta)\theta)\mu+\mu^2}{(\delta+\pi(-1+\vartheta)\theta)(\gamma+\mu)(\beta+\delta+\mu)} \\
T^* &= \frac{\beta\Lambda(\gamma(\beta+\delta+\pi(-1+\vartheta)\theta)+\beta+\gamma+\delta-\pi(-1+\alpha)(-1+\vartheta)\theta)\mu+\mu^2}{(\delta+\pi(-1+\vartheta)\theta)(\gamma+\mu)(\beta+\delta+\mu)\mu}
\end{aligned} \tag{2.7}$$

since N is given by

$$N = \frac{\pi(-1+\vartheta)\theta(\alpha\Lambda\mu+I(\beta+\mu)(\gamma+\mu)}{\mu(\gamma+\mu)(\beta+\delta+\pi(-1+\vartheta)\theta+\mu}$$

Upon simplification I^* becomes

$$I^* = \frac{(R_0-1)[(\Lambda(\beta+\delta+\mu)(\frac{\gamma}{\beta+\delta}+\frac{\mu}{(\beta+\gamma+\delta)(1-\alpha)})-\frac{\Lambda\mu^2}{\beta+\delta+\mu}]}{\delta-\pi\theta(1-\vartheta)(\gamma+\mu)} \tag{2.8}$$

$I^* > 0$ provided that $R_0 > 1$ with $(\Lambda(\beta+\delta+\mu)(\frac{\gamma}{\beta+\delta}+\frac{\mu}{(\beta+\gamma+\delta)(1-\alpha)}) > \frac{\Lambda\mu^2}{\beta+\delta+\mu}$ and $\delta > \pi\theta(1-\vartheta)(\gamma+\mu)$, implying that death due to typhoid fever disease decreases with an increase in protection.

2.2 Global stability of the disease-free equilibrium (DFE)

For global stability of the DFE, the technique used by [9] was employed. There are two conditions that if met, the global asymptotic stability of the disease free equilibrium point is guaranteed. Equation (2.1) may be written in the form

$$\begin{aligned} \frac{dX}{dt} &= K(X,Z) \\ \frac{dZ}{dt} &= G(X,Z), G(X,0)=0 \end{aligned} \tag{2.9}$$

where $X \in \mathbb{R}^3$ and $X = \{S, P, T\}$ denotes the number of uninfected individuals and $Z \in \mathbb{R}^1$ where $Z = \{I\}$ denotes the number of infected individuals . $E^O = (\frac{\Lambda}{\mu}, 0, 0, 0)$ denotes the disease free equilibrium point of this system where

$$X^* = (\frac{\Lambda}{\mu})$$

conditions (2.10) may be met to guarantee global asymptotic stability

$$\begin{aligned} \frac{dX}{dt} &= K(X,0), X^* \text{is globally asymptotic stable} \\ G(X,Z) &= AZ - \widehat{G}(X,Z), \widehat{G}(X,Z) \geq 0 \forall (X,Z) \in \Gamma \end{aligned} \tag{2.10}$$

where $A = D_z G(X^*, 0)$ is an M matrix and Γ is the region where the model has biological meaning.

Theorem

If system (2.1) satisfies conditions (2.10), then the fixed point $E^0 = (X^*, 0, 0, 0)$ is a globally asymptotically stable equilibrium of the system (2.1) provided that $R_0 < 1$.

Proof
Consider $K(X,0) = (\Lambda - \mu S)$ and $G(X,Z) = AZ - \widehat{G}(X,Z)$ where

$$A = -(\delta + \beta + \mu). \tag{2.11}$$

and

$$\hat{G}(X,Z) = -\frac{\pi\theta(1-\vartheta)I}{N}S. \tag{2.12}$$

Since all the conditions in Equation (2.10) are not satisfied because $\hat{G}(X,Z) < 0$, the DFE E^0 may not be globally asymptotically stable, implying that we anticipate an outbreak when particular conditions which favour the outbreak of the disease are prevailing.

2.3 Global stability of the endemic equilibrium point (EEP)

The global stability of the endemic equilibrium may be obtained by means of Lyapunov's direct method and LaSalle's invariance principle [7]. Consider the non-linear Lyapunov function

$$L : (S, P, I, T) \in \Gamma \subset \mathbb{R}^4_+ : S, P, I, T > 0 \tag{2.13}$$

where

$$\begin{aligned} L : (S,P,I,T) = \lambda(S - S^* - S^* \log \frac{S}{S^*}) + \lambda(P - P^* - P^* \log \frac{P}{P^*}) + \\ + \lambda(I - I^* - I^* \log \frac{I}{I^*}) + \lambda(T - T^* - T^* \log \frac{T}{T^*}) \end{aligned} \tag{2.14}$$

where L is C^1 in the interior of Γ. E^* is the global minimum of L on Γ and $L:(S,P,I,R)=0$. The time derivative of L is given by

$$\frac{dL}{dt}=\dot{L}=\lambda(1-\frac{S^*}{S})\frac{dS}{dt}+\lambda(1-\frac{P^*}{P})\frac{dP}{dt}+\lambda(1-\frac{I^*}{I})\frac{dI}{dt}+\lambda(1-\frac{T^*}{T})\frac{dT}{dt} \tag{2.15}$$

with the derivatives of Equation S,P,I,T defined in Equation (2.1) and by using $\alpha\Lambda=(\gamma+\mu)P^*$, $(1-\alpha)\Lambda=-\gamma P^*+(\lambda+\mu)S^*$, $\lambda S^*=(\delta+\beta+\mu)I^*$, $\beta I=\mu T^*$ into Equation (2.15) we obtain

$$\begin{aligned}\dot{L}=-\lambda(\frac{S-S^*}{S})[(\mu+\lambda)(S-S^*)+\gamma(P-P^*)]-\lambda(\frac{P-P^*}{P})[(\gamma+\mu)(P-P^*)]\\ -\lambda(\frac{I-I^*}{I})[(\mu+\beta)(I-I^*)]-\lambda(\frac{T-T^*}{T})[(\mu)(T-T^*)]\end{aligned} \tag{2.16}$$

Hence $\dot{L}<0$. We see that $\dot{L}=0$ iff $S=S^*,P=P^*,I=I^*$ and $T=T^*$. Thus the largest compact invariant set in $\{S,P,I,T\}\in\Gamma:\dot{L}=0$ is the Singleton E^*, where E^* is the endemic equilibrium. Thus E^* is globally asymptotically stable in the interior of the region Γ. Mathematically, we have shown that protection produces desired results in terms of disease intervention.

3 Discussion

The global stability of the disease free equilibrium and the disease endemic equilibrium point was carried out. From the analysis, the DFE E^0 is not globally asymptotically stable, implying that we anticipate an outbreak when particular conditions which favour the outbreak of the disease are prevailing, or when people stop embracing protective measures. By use of a suitable logarithmic Lyapunov function, the endemic state is shown to be globally asymptotically stable. This implies that disease transmission levels can be kept quiet low or manageable with minimal deaths in the presence of protection.

4 Conclusion

We conclude that typhoid fever disease, can be effectively controlled from spreading in a given population by embracing protective measures such as drinking clean water, improved sanitation and receiving proper treatment for those already infected among others. Death due to typhoid fever disease decreases with an increase in protection because there will be a decrease in the number of infected individuals.

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Competing Interests

Author has declared that no competing interests exist.

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