# Existence, Uniqueness and Blow-up Result of Solutions for an Evolution $p(x)-$laplacian Equation

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### *Author's contribution*

*The sole author designed, analyzed and interpreted and prepared the manuscript.*

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## Abstract

In this paper we are investigate in the evolution equation p(x)- laplacian with the initial boundary value question. We translate the parabolic equation into the elliptic equation by using a finite difference method, and then the existence and uniqueness solution are obtained. The blow-up property is shown, by using the energy method. We perform, using Matlab (Ode45 subroutine), some numerical experiments just to illustrate our general results.

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# 1  INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded Lipshitz domain and $0 < T < \infty$. It will be assumed throughout this paper that $p(x)$ is continuous function defined in $\overline{\Omega}$ with logarithmic module of continuity :

$$2 < p^- = \inf_{\overline{\Omega}} p(x) \leq p(x) \leq p^+ = \sup_{\overline{\Omega}} p(x) < \infty,$$

$$|p(x) - p(y)| \leq -\frac{C}{\log |x-y|}, \text{ for any } x, y \in \Omega$$

$$\text{with } |x - y| < \frac{1}{2}. \tag{1.1}$$

In this paper, we consider the following $p(x) - \text{laplacian}$ equation :

$$\begin{cases} a(x)\frac{\partial u}{\partial t} - \Delta_{p(x)} u = f(x, u), & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $p(x) \in C(\overline{\Omega})$ is a function. The operator $-\Delta_{p(x)} u = -\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$ is called $p(x)-$Laplacian, which will be reduced to the $p - Laplacian$ when $p(x) = p$ a constant.

In the case when $a(x) = 1$ and $p(x)$ is constant, there have been many results about the existence, uniqueness, and some other properties of the solutions to problem (1.2), we refer to the readers to the bibliography given in $[1, 2, 3]$ (see also Refs.$[4, 5, 6, 7]$), and the references therein. Recently, $[8]$ study the equation the $p(x) - \text{laplacian}$ equation

$$a(x)\frac{\partial u}{\partial t} = \operatorname{div}(u^{m-1} |Du|^{\lambda-1} Du),$$

where $\lambda > 0$, $m + \lambda - 2 > 0$ and $a(x)$ is a positive continuous function. They examine under which conditions on behavior of $a(x)$, corresponding nonnegative solutions of the Cauchy problems possess the finite speed of propagations or interface blow-up phenomena.

In recent years, the research of nonlinear problems with variable exponent growth conditions has been an interesting topic. $p(\cdot)$-growth problems can be regarded as a kind of nonstandard growth problems and these problems possess very complicated nonlinearities, for instance, the $p(x)$-Laplacian operator is inhomogeneous. And these problems have many important applications in nonlinear elastic, electrorheological fluids and image restoration. The reader can find in $[9, 10, 11, 12, 13, 14]$ (see also Refs.$[15, 16, 17, 18, 19, 20, 21]$), several models in mathematical physics where this class of problem appears.

In this paper, we consider the existence and uniqueness for the problem of the type (1.2) under some assumptions. The proof consists of two steps. First, we prove that the approximating problem admits a global solution; then we do some uniform estimates for these solutions. We mainly use skills of inequality estimation and the method of approximation solutions. By a standard limiting process, we obtain the existence to problem of the type (1.2).

The outline of this paper is the following: In Section 2, we introduce some basic Lebesgue and Sobolev spaces and state our main theorems. In Section 3, we give the existence and uniqueness of weak solutions. In section 4, the blow-up results will be proved. In section 5, we show some numerical experiments.

# 2  BASIC SPACES AND THE MAIN RESULTS

To consider problems with variable exponents, one needs the basic theory of spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For the convenience of readers, let us review them briefly here. The détails and more properties of variable-exponent Lebesgue-Sobolev spaces can be found in $[2, 22]$.

Let $p(x) \in C(\overline{\Omega})$. When $p^- > 1$, one can introduce the variable-exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; \ u \text{ is measurable and } \int_\Omega |u|^{p(x)} \, dx < \infty \right\},$$

endowed with the Luxemburg norm.

$$\|w\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{w}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}.$$

The conjugate space is $L^{q(x)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \ \forall x \in \overline{\Omega}$.

As in the case of a constant exponent, set

$$W^{1,p(x)}(\Omega) = \left\{ u(x) \in L^{p(x)}(\Omega) : \ |\nabla u|^{p(x)} \in L^1(\Omega) \right\}.$$

endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \,.$$

Similarly we also denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $\left( W_0^{1,p(x)}(\Omega) \right)'$ is the dual of $W_0^{1,p(x)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$.

In Propositions 2.1-2.3, we describe some results about the Luxembourg norm.

**Proposition 2.1.** *([2, 22]) (1) The space $\left( L^{p(x)}(\Omega), \|.\|_{p(x)} \right)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \ \forall x \in \overline{\Omega}$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality :*

$$\left| \int_\Omega uv dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(x)} \|v\|_{q(x)} \leq 2 \|u\|_{p(x)} \|v\|_{q(x)} \,.$$

*(2) If $r_1(x) \leq r_2(x)$ for any $x \in \Omega$, the imbedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$ is continuous, the norm of the imbedding does not exceed $|\Omega| + 1$.*

**Proposition 2.2.** [23] *If we denote*

$$\rho(w) = \int\limits_\Omega |w|^{r(x)} \, dx, \ \forall w \in L^{r(x)}(\Omega),$$

*then*

(1) $|w|_{r(x)} < 1 \ (= 1; > 1) \Leftrightarrow \rho(w) < 1 \ (= 1; > 1);$

(2) $|w|_{r(x)} > 1 \Rightarrow |w|_{r(x)}^{r^-} \leq \rho(w) \leq |w|_{r(x)}^{r^+} \,; \ |w|_{r(x)} < 1 \Rightarrow |w|_{r(x)}^{r^+} \leq \rho(w) \leq |w|_{r(x)}^{r^-} \,;$

(3) $|w|_{r(x)} \to 0 \Leftrightarrow \rho(w) \to 0; \ |w|_{r(x)} \to \infty \Leftrightarrow \rho(w) \to \infty.$

**Proposition 2.3.** [22] *For $u \in W_0^{1,p(x)}(\Omega)$, there exists a constant $C = C(p, |\Omega|) > 0$, such that*

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)} \,,$$

*This implies that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms of $W_0^{1,p(x)}(\Omega)$.*

Problem (1.2) does not admit classical solutions in general. So, we introduce weak solutions in the following sence.

**Definition 2.1.** A function $u$ is said to be a weak solution of Problem (1.2), if the following conditions are satisfied :

(1) $u \in L^\infty(0, T, W_0^{1,p(x)}(\Omega)) \cap C(0, T; L^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0, T, W_0^{-1,p'(x)}(\Omega))$ such that :

(2) For any $\phi \in C_0^\infty(Q_T)$ and $Q_T = \Omega \times (0, T)$

$$\int\limits_0^T \int\limits_\Omega (a(x)u\phi_t - |\nabla u|^{p(x)-2}\, \nabla u \nabla \phi - f(x, u)\phi)dxdt = 0$$

(3) $u(x, 0) = u_0(x)$.

In the study of the global existence of solutions, we need the following hypotheses (H):

(H1) $u_0 \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$

(H2) $0 < C \le a(x) \in L^\infty(\Omega)$

(H3) $f(x, s) \in C^1(\overline{\Omega} \times \mathbb{R})$.


# 3  MAIN RESULTS

In this paper, we shall denote by $c, C_i$ differents constants, depending on $p_i(x), T, \Omega$, but not on $n$, which may vary from line to line. Sometimes we shall refer to a constant depending on specific parameters $C_i(T)$, etc.

Our main existence result is the following:

**Theorem 3.1.** *Let (H1)-(H3) hold. Then problem (1.2) admits a unique solution $u \in C([0, T]; L^2(\Omega))$. Moreover, the mapping $u_0 \to u(t)$ is continuous in $L^2(\Omega)$.*

*Proof of the main results.* □


## 3.1  Existence

We will semi-discrete (1.2) in time and solve the corresponding elliptic problem. Based on the semi-discrete problem, we construct the corresponding approximate solutions. The key procedure is to establish necessary a priori estimates for finding the limit of the approximate solutions via a compactess argument.

We first consider the discrete scheme (3.1)

$$a(x)\frac{u^n - u^{n-1}}{\tau} - \Delta_{p(x)}u^n = f(x, u^{n-1}) \quad \text{in } \Omega, \tag{3.1}$$

$$u^n = 0 \quad \text{on } \partial\Omega,$$

$$u^0 = u_0 \quad \text{in } \Omega,$$

where $N\tau = T$ and $T$ is a fixed positive real, and $1 \le n \le N$.

**Lemma 3.2.** *For any fixed n, if $u^{n-1} \in W_0^{1,p(x)}(\Omega) \cap \mathbf{L}^\infty(\Omega)$, Problem (3.1) admits a weak solution $u^n \in W_0^{1,p(x)}(\Omega) \cap \mathbf{L}^\infty(\Omega)$.*

*Proof.* On the space $W_0^{1,p(x)}(\Omega)$, we consider the functional

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{1}{2\tau} \int_\Omega a(x) |u|^2 \, dx - \int_\Omega gu dx.$$

where $g \in \mathbf{L}^\infty(\Omega)$ is a known function. Using Young's inequality and Proposition 2.1, there exist constants $C_1, C_2 > 0$, such that

$$\Phi(u) \geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} \, dx - C_2 \|g\|_{L^2}^2 \geq \frac{1}{p^+} \|u\|_{1,p(x)}^{p^-} - C_2 \|g\|_{L^2}^2 \; ;$$

hence $\Phi(u) \to \infty$, as $\|u\|_{1,p(x)} \to +\infty$. Since the norm is lower semi-continuous and $\int_\Omega gu dx$ is continuous functional, $\Phi(u)$ is weakly lower semi-continuous on $W_0^{1,p(x)}(\Omega)$ and satisfy the coercive condition. From [24] we conclude that there exists $u^* \in W_0^{1,p(x)}(\Omega)$, such that

$$\Phi(u^*) = \inf_{u \in W_0^{1,p(x)}(\Omega)} \Phi(u)$$

and $u^*$ is the weak solutions of the Euler equation corresponding to $\Phi(u)$,

$$a(x)\frac{u}{\tau} - \triangle_{p(x)} u = g.$$

Choosing $g = f(x, u^{n-1}) + a(x)\frac{1}{\tau} u^{n-1}$, we obtain a weak solution $u^n$ of (3.1).

$$a(x)\frac{u^1 - u^0}{\tau} - \Delta_{p(x)} u^1 = f(x, u^0). \tag{3.2}$$

Since $|f(x, u_0)| \leq M$, we may prove by induction that (3.1) has a solution $u^n$ in $\mathbf{L}^\infty(\Omega)$. We put $u^1 := w$ and for any integer $k > 0$, we may take $(w - M\tau)_+^k$ as a test function in (3.2) to get

$$\int_\Omega \frac{1}{\tau}(w - M\tau)_+^{k+1} dx + k \int_\Omega \left| \nabla(w - M\tau)_+^{p(x)} \right| (w - M\tau)_+^{k-1} dx$$

$$= \int_\Omega \frac{1}{\tau}(w - M\tau)_+^k u^0 dx + \int_\Omega f(x, u^0)(w - M\tau)_+^k dx$$

By the Hôlder inequality and $|f(x, u_0)| \leq M$, we have

$$\int_\Omega (w - M\tau)_+^{k+1} dx \leq \left( \int_\Omega (w - M\tau)_+^{k+1} \left( u^0 + M\tau \right) dx \right)$$

$$\leq \left( \int_\Omega (w - M\tau)_+^{k+1} dx \right)^{\frac{k}{k+1}} \left( \int_\Omega^{k+1} \left( u^0 + M\tau \right)^{k+1} dx \right)^{\frac{1}{k+1}}$$

We deduce $\|(w - M\tau)_+\|_{L^{k+1}(\Omega)} \leq \|u^0 + M\tau\|_{L^{k+1}(\Omega)}$.
Letting $k \to \infty$, we get $(w)_+ \leq \|u^0\|_{L^\infty(\Omega)} + 2M\tau$. Consider $-w$, we get easily that
$(w)_- \geq - \|u^0\|_{L^\infty(\Omega)} - 2M\tau$, i.e. $\|u^1\|_{L^\infty(\Omega)} \leq \|u^0\|_{L^\infty(\Omega)} + 2M\tau$ and if we choose $\tau$ such that
$\tau \leq \frac{1}{2M}$, we obtain $u^n \in \mathbf{L}^\infty(\Omega)..$
This completes the proof of lemma 3.3.
Now, we define the approximate solutions as $(u)_\tau, (\widetilde{u})_\tau$ set by : $for$ all $n \in \{1, ..., N\}$.

$$\forall t \in [(n-1)\tau, n\tau] \quad \begin{cases} u_\tau(t) = u^n, \\ \\ \widetilde{u}_\tau(t) = \frac{(t-(n-1)\tau)}{\tau}(u^n - u^{n-1}) + u^{n-1} \end{cases}$$

are well defined and satisfied in addition

$$\frac{\partial \widetilde{u}_\tau}{\partial t} - \Delta_{p(x)} u_\tau = f(x, u_\tau(. - \tau)) \tag{3.3}$$

We first establish some energy estimates of $u_\tau, \widetilde{u}_\tau$.

We need several lemmas to complete the proof of Theorem 3.2.                                                □

**Lemma 3.3.** *There exists a positive constant $C(T, u_0)$ such that, for all $n = 1, \dots, N$*

$$u^n \in L^\infty(0, T; L^\infty(\Omega)) \tag{3.4}$$

$$u_\tau, \widetilde{u}_\tau \text{ are bounded in } L^p(0, T; W_0^{1,p(x)}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \tag{3.5}$$

$$\frac{\partial \widetilde{u}_\tau}{\partial t} \text{ is bounded in } L^2(Q_T) \tag{3.6}$$

*and*

$$u_\tau, \widetilde{u}_\tau \text{ are bounded in } L^\infty(0, T; W_0^{1,p}(\Omega)). \tag{3.7}$$

*Proof.* **(a)** By lemma 3.2, for any $n \in N$, $u^n$ is bounded; whence (3.4)
   **(b)** Multiplying (3.1) by $\tau u^n$, summing from $n = 1$ to $N$ and integrating over $\Omega$ we obtain

$$\tau \sum_{n=1}^N \int_\Omega a(x) \left( \frac{u^n - u^{n-1}}{\tau} \right) u^n dx + \tau \sum_{n=1}^N \int_\Omega |\nabla u^n|^{p(x)} dx = \tau \sum_{n=1}^N \int_\Omega f(x, u^{n-1}) u^n dx. \tag{3.8}$$

By Young Inequality, for $\epsilon > 0$ small, there exists $C_\epsilon(T)$ such that

$$\tau \sum_{n=1}^N \int_\Omega f(x, u^{n-1}) u^n dx \leq \epsilon \tau \sum_{n=1}^N \int_\Omega |\nabla u^n|^{p(x)} dx + C_\epsilon(T). \tag{3.9}$$

With the aid of the identity $2\alpha(\alpha - \beta) = \alpha^2 - \beta^2 + (\alpha - \beta)^2$, we get

$$\tau \sum_{n=1}^N \int_\Omega a(x) \left( \frac{u^n - u^{n-1}}{\tau} \right) u^n dx = \frac{1}{2} \sum_{n=1}^N \int_\Omega a(x) \left( |u^n|^2 - |u^{n-1}|^2 + |u^n - u^{n-1}|^2 \right) dx$$

$$= \frac{1}{2} \sum_{n=1}^N \int_\Omega a(x) \left( |u^n|^2 - |u^{n-1}|^2 \right) dx + \frac{1}{2} \int_\Omega a(x) \left| u^N \right|^2 dx - \frac{1}{2} \int_\Omega a(x) |u_0|^2 dx.$$

With the above estimates, we get (3.5).

**(c)** Multiplying the equation (3.1) by $u^n - u^{n-1}$ and summing from $n = 1$ to $N$, wet get

$$\tau \sum_{n=1}^N \int_\Omega a(x) \left( \frac{u^n - u^{n-1}}{\tau} \right)^2 dx + \sum_{n=1}^N \int_\Omega |\nabla u^n|^{p(x)-2} \nabla u^n . \nabla \left( u^n - u^{n-1} \right) dx$$

$$= \sum_{n=1}^N \int_\Omega f(x, u^{n-1}) \left( u^n - u^{n-1} \right) dx.$$

By Young Inequality, we get

$$\sum_{n=1}^{N} \int_{\Omega} f(x, u^{n-1}) \left( u^n - u^{n-1} \right) dx \leq C_{\epsilon}(T) + \frac{\tau}{2} \sum_{n=1}^{N} \int_{\Omega} \left( \frac{u^n - u^{n-1}}{\tau} \right)^2 dx \qquad (3.10)$$

From the convexity of the expression $\int_{\Omega} |\nabla w|^{p(x)} dx$, we get the following inequality:

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u^n|^{p(x)} dx - \int_{\Omega} \frac{1}{p(x)} |\nabla u^{n-1}|^{p(x)} dx \leq \int_{\Omega} |\nabla u^n|^{p(x)-2} \nabla u^n . \nabla \left( u^n - u^{n-1} \right) dx \quad (3.11)$$

which imply with (3.9) and (3.10) that

$$\frac{\tau}{2} \sum_{n=1}^{N} \int_{\Omega} a(x) \left( \frac{u^n - u^{n-1}}{\tau} \right)^2 dx + \int_{\Omega} \frac{1}{p(x)} \left| \nabla u^N \right|^{p(x)_i} dx \leq C. \qquad (3.12)$$

Thus we obtain

$$\left( \frac{\partial \widetilde{u}_{\tau}}{\partial t} \right)_{\tau} \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \tau, \qquad (3.13)$$

$$(u_{\tau}) \text{ and } (\widetilde{u}_{\tau}) \text{ are bounded in } L^{\infty}(0, T; W_0^{1,p(x)}(\Omega)) \text{ uniformly in } \tau. \qquad (3.15)$$

By lemma 3.4, there exists $M_i > 0$ independent of $\tau$ such that

$$\|u_{\tau} - \widetilde{u}_{\tau}\|_{L^{\infty}(0,T;L^2(\Omega))} \leq \max_{1 \leq n \leq N} \left\| u^n - u^{n-1} \right\|_{L^2(\Omega)} \leq M\sqrt{\tau}. \qquad (3.16)$$

Therefore, taking $\tau \to 0^+$, and up to subsequence, wet get that there exists $u, v \in L^{\infty}(0, T; W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega))$ such that $\frac{\partial u}{\partial t} \in L^2(Q_T)$,

$$u_{\tau} \overset{*}{\to} u \text{ in } L^{\infty}(0, T; W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)) \text{ and } \widetilde{u}_{\tau} \overset{*}{\to} v \text{ in } L^{\infty}(0, T; W_0^{1,p(x)}(\Omega) \cap L^{\infty}(\Omega)) \quad (3.17)$$

$$\frac{\partial \widetilde{u}_{\tau}}{\partial t} \overset{\text{weak}}{\to} \frac{\partial u}{\partial t} \text{ in } L^2(Q_T). \qquad (3.18)$$

From (3.16), it follows that $u = v$. From (3.17), we get that

$$u_{\tau}, \widetilde{u}_{\tau} \to u \text{ in } L^q(0, T; W_0^{1,p(x)}(\Omega)), \forall q \geq 1. \qquad (3.19)$$

By Aubin-Simon's compactness results [25], we have

$$\widetilde{u}_{\tau} \to u \in C(0, T; L^2(\Omega)). \qquad (3.20)$$

Now, multiplying (3.1) by $u_{\tau} - u$ and using (3.11) and (3.16), we get by straightforward calculations :

$$\int_0^T \int_{\Omega} a(x) \left( \frac{\partial \widetilde{u}_{\tau}}{\partial t} - \frac{\partial u}{\partial t} \right) (\widetilde{u}_{\tau} - u) dx dt - \int_0^T < \triangle_{p(x)} u_{\tau}, u_{\tau} - u > dt$$

$$= \int_0^T \int_{\Omega} f(x, (u_{\tau}(. - \tau)) dx dt + o_{\tau}(1),$$

where $o_\tau(1) \to 0$ as $\tau \to 0^+$.

Thus, we get that

$$\frac{1}{2} \int_\Omega a(x) \left| \widetilde{u}_\tau(T) - u(T) \right|^2 dx - \int_0^T < \triangle_{p(x)} u_\tau - \triangle_{p(x)} u, u_\tau - u > dt$$

$$\leq \int_0^T \int_\Omega f(x, u_\tau(. - \tau)) dx dt + o_\tau(1), \tag{3.21}$$

and from (3.17) we have thus,

$$u_\tau \to u \ in \ L^{p(x)}(0, T; W_0^{1,p(x)}(\Omega)), \ as \ \tau \to 0^+$$

and consequently by the same as that in [8]

$$\triangle_{p(x)} u_\tau \to \triangle_{p(x)} u \ in \ L^{p(x)'}(0, T; W^{-1,p(x)'}(\Omega)).$$

Therefore, $u$ satisfies **(**1.2**)**. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

## 3.2 Uniqueness

**Theorem 3.4.** *Let (H1) to (H3) be satisfied. Then problem (1.2) has a unique solution $u$ in $Q_T$.*

*Proof.* Let $u$ and $v$ be solutions of (1.2**)**, we have :

$$\int_0^T \int_\Omega a(x) \frac{\partial (u - v)}{\partial t} (u - v) \, dx dt - \int_0^T < \triangle_{p(x)} u - \triangle_{p(x)} v, u - v > dt$$

$$= \int_0^T \int_\Omega (f(x, u) - f(x, v)) (u - v) \, dx dt,$$

Since $f(x, .)$ is locally lipschitz uniformly in $\Omega$, the difference $w = u - v$ satisfies

$$\frac{1}{2} C \left| w \right|_{L^2(\Omega)}^2 + \int_0^T < \triangle_{p(x)} u - \triangle_{p(x)} v, w > dt \leq c \int_0^T \int_\Omega \left| w \right|^2 dt,$$

we observe that $w \to -\triangle_{p(x)} w$ is monotone from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p(x)'}(\Omega)$

$$\left| w \right|^2 \leq 2c \int_0^T \left| w \right|^2 dt. \tag{3.22}$$

We finally deduce from Gronwall's lemma,

$$\left| w \right|^2 \leq \left| w(0) \right|^2 \exp(2cT), \forall t \in (0, T).$$

Thus, we deduce that $u = v$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

# 4 BLOW-UP RESULTS

In this section, we shall investigate the blow-up properties of solutions to problem (1.2), using energy methods. To this end, we consider the following hypotheses on the data.

(H4) $u_0 \in W_0^{1,p(.)}(\Omega) \cap L^{p(.)}(\Omega)$ such that $\int_\Omega F(u_0(x)) dx - \int_\Omega \frac{1}{p(x)} \left| \nabla u_0 \right|^{p(x)} dx \geq 0$.

(H5) $f(x, u) = h(u)$ and and $h$ is such that :

$$|u|^\alpha \le \alpha F(u) \le u f(u), \quad \alpha > \max(p^+, 2).$$ (4.0)

Throughout this section, we define for $t \ge 0$,

$$E(t) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_\Omega F(u(x,t)) dx$$

where $F(t) = \int_0^t f(s) ds$.

**Theorem 4.1.** *Let (H1) to (H5) be satisfied then the solutions of Problem (1.2) blow up in finite time, namely, there exists a $T^* < \infty$ such that $\|u(., t)\|_{\infty, \Omega} \to \infty$ as $t \to T^*$.*

*Proof.* Multiplying the equation (1.2) by $\frac{\partial u}{\partial t}$, integrating by parts, we have

$$g'(t) = \int_\Omega a(x) u \frac{\partial u}{\partial t} dx = -\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega u f(u) dx$$

$$= -\int_\Omega p(x) \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega u f(u) dx$$

$$\ge -p^+ \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega u f(u) dx$$

$$\ge -p^+ \left( E(t) + \int_\Omega F(u(x,t)) dx \right) + \int_\Omega u f(u) dx$$

$$\ge \int_\Omega u f(u) dx - p^+ \int_\Omega F(u(x,t)) dx$$

$$\ge \int_\Omega u f(u) dx - p^+ \int_\Omega F(u(x,t)) dx$$

$$\ge \left( \frac{\alpha - p^+}{\alpha} \right) \int_\Omega |u|^\alpha dx.$$ (4.1)

By using Hölder's inequality, we have

$$g^{\frac{\alpha}{2}}(t) \le \left( \frac{\|a\|_\infty}{2} \right)^{\frac{\alpha}{2}} |\Omega|^{\frac{\alpha-2}{2}} \int_\Omega |u|^\alpha dx.$$ (4.2)

Thus, it is deduced by combining (4.1) and (4.2) that

$$g'(t) \ge k g^{\frac{\alpha}{2}}(t),$$

where $k = \left( \frac{2}{\|a\|_\infty} \right)^{\frac{2}{\alpha}} (1 - \frac{p^+}{\alpha}) |\Omega|^{\frac{2-\alpha}{2}} > 0$.

A direct integration of the above inequality over $(0, t)$ then yields

$$g^{\frac{\alpha}{2}-1}(t) \ge \frac{1}{g^{1-\frac{\alpha}{2}}(0) - k(\frac{\alpha}{2} - 1)t},$$

□

which implies that g(t) blows up at a finite time $T^* \leq g^{1-\frac{\alpha}{2}}(0)/(k(\frac{\alpha}{2}-1))$, and so does $u$.

# 5   NUMERICAL COMPUTATION

We solve the problem (1.2) with the term $f = \lambda u^\alpha(x,t)$, where $\alpha > 1$ and the domain $\Omega$ is just the real line, that is $\Omega = [0,1]$ . The problem becomes :

$$\begin{cases} a(x)u_t(x,t) = \left(|u_x(x,t)|^{p(x)-2} u_x(x,t)\right)_x + \lambda u^\alpha(x,t), \quad x \in [0,1], t > 0 \\ \qquad\qquad u(x,0) = u_0(x) \quad x \in [0,1], \\ \qquad\qquad u(x,t) = 0 \quad x \in \partial\Omega, t > 0. \end{cases},$$

For the special discretization, we choose a uniform mesh $D_h = \{x_i : 0 = x_0 < x_1 < ... < x_{M+1} = 1\}$ (with $x_i = ih$) on $\Omega$ and replace $\left(|u_x|^{p(x)-2} u_x\right)_x ((x_i,t)\ (1 \leq i \leq M)$ by the standard central difference approximation.

We use the function $10 \sin(\pi x)$ as the initial function, we get the following system of Ode, for $U_i(t) \approx u(x_i,t)\ (1 \leq i \leq M)$ :

$$\text{(S)}\begin{cases} a(x_i)U_i'(t) = h^{-p(x_i)}A(U_i(t)) + \lambda U_i^\alpha(t),\ 1 \leq i \leq M, \\ \qquad\qquad U_0(t) = U_{M+1}(t), \\ \qquad\qquad U_i(t) = 10\sin(\pi x_i). \end{cases},$$

where $q_i = p(x_i) - 2$ and $A(U_i(t)) = |U_{i+1}(t) - U_i(t)|^{q_i}(U_{i+1}(t) - U_i(t)) - |U_i(t) - U_{i-1}(t)|^{q_i}(U_i(t) - U_{i-1}(t))$

We solve the problem (S) with the term MATLAB solvers ode45 and we illustrate our previous results with some numerical experiments which show some of the properties observed for the numerical solutions. In all cases we take the initial data $u_0(x) = 10.\sin(\pi.x), x \in [0,1]$. The other parameters are specified in the graphs: (Figs. 1, 2, 3, 4).
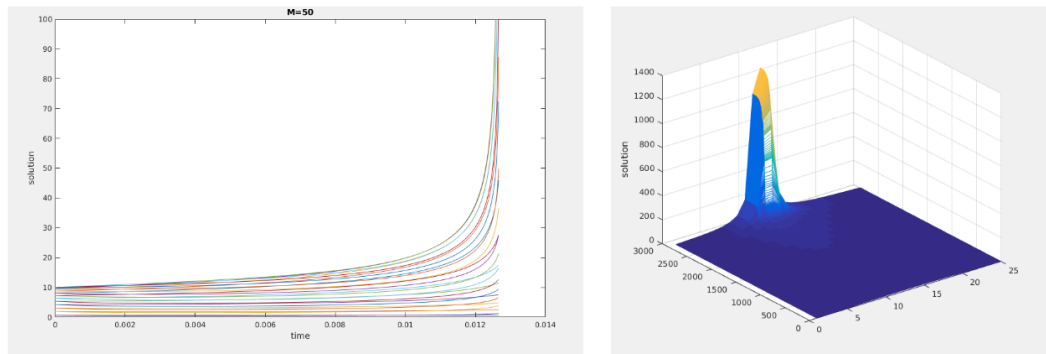


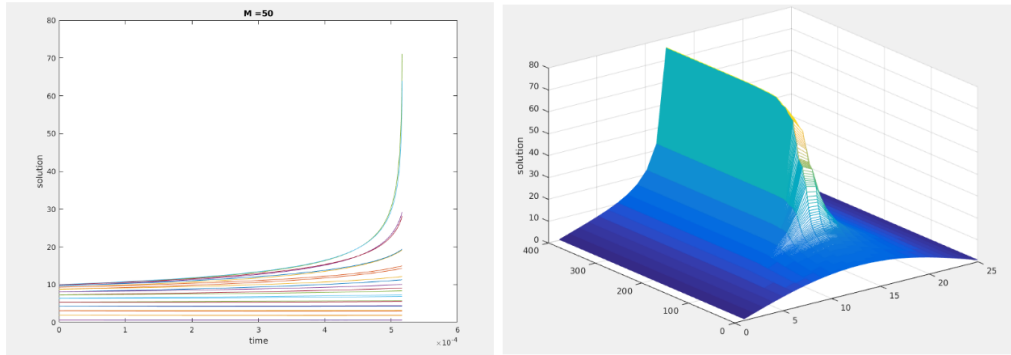**Fig. 1. Numerical solution of (1.2) with** $a(x) = 1 + x$**,** $p(x) = 2 + x$ **and** $\alpha = 3$

**Fig. 2. Numerical solution of (1.2) with** $a(x) = 1 + x$**,** $p(x) = 2 + x$ **and** $\alpha = 4$
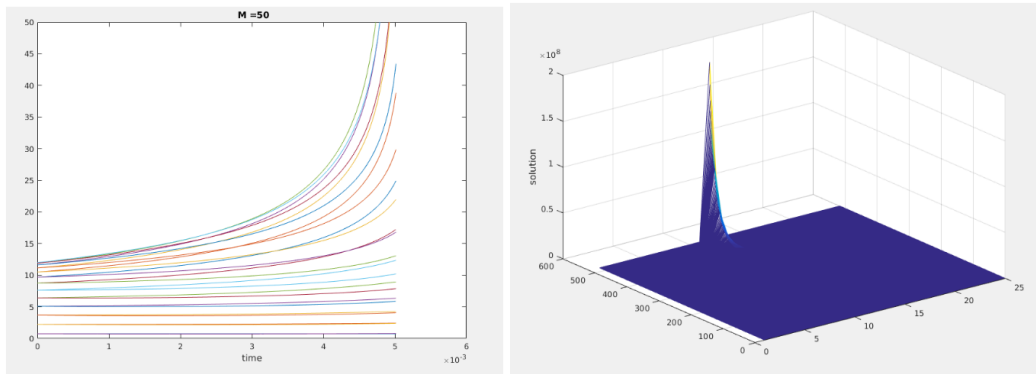


**Fig. 3. Numerical solution of (1.2) with** $a(x) = 1 + x^2$**,** $p(x) = 2 + \frac{1}{2}.x$ **and** $\alpha = 3$
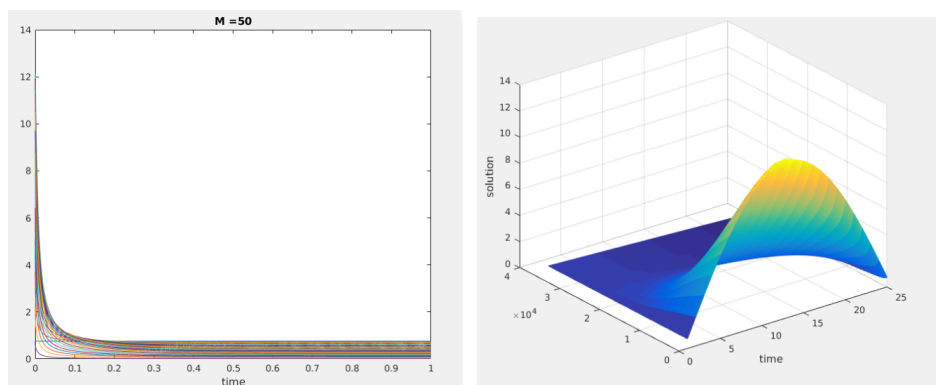


**Fig. 4. Numerical solution of (1.2) with** $a(x) = 1 + x^2$**,** $p(x) = 2.5 + x$ **and** $\alpha = 3$

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# 6 CONCLUSION

In paper [8], the author studies the problem (1.2) with the operator $\mathrm{div}(u^{m-1} |Du|^{\lambda-1} Du)$ and the source term null, our main contribution is to generalize this work to the p(x)-laplacian operator with the source term f satisfying the conditions of type (4.0). We have carried out several numerical examples in one dimension with variable a(x) and p(x). In equation (S), the non-linear term $f(x, u)$ describes the non-linear source in the diffusion process. We describe our results in the one-dimensional case. Of course, most physical problems are described in two or three dimensions. The extension to several space dimensions is straightforward.

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# COMPETING INTERESTS

Author has declared that no competing interests exist.

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