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EXACT SOLUTIONS OF FRACTIONAL MAXWELL FLUID BETWEEN TWO CYLINDERS

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Abstract. In this paper the velocity field and the adequate shear stress corresponding to the rotational flow of a fractional Maxwell fluid, between two infinite coaxial circular cylinders with inner cylinder is at rest and outer is moving, are determined by applying the Laplace and finite Hankel transforms. The solutions that have been obtained are presented in terms of generalized G functions. The expressions for the velocity field and the shear stress are in the most simplified form. Moreover, these solutions satisfy both the governing differential equation and all imposed initial and boundary conditions. The corresponding solutions for ordinary Maxwell and Newtonian fluids are recovered as limiting cases of general solutions.

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1. Introduction

Fluid has necessarily become part and parcel of daily human life. Take a look at any material of some importance in your life and you will eventually encounter fluid of some type. Air, gases, water, and liquids of various types all fall under the definition of a fluid. Consequently, basic concepts and principles of fluid mechanics are essentially important for us. Any system in which fluid is a working medium will be subject to analysis through the help of fluid mechanics. Principles of fluid mechanics are applied on the designs of almost all means of transportation. We can safely include the subsonic and supersonic aircrafts,

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hovercrafts, surface ships, submarines and other automobiles. Aerodynamic designs which had previously been confined for racing cars and boats only are now a subject of importance for all automobile manufacturers. Rockets, space flights and others having propulsion systems are based on the fluid mechanics principles. It is a common practice now to undertake prior studies on the aerodynamic forces around buildings and structures. The designs of some devices like pumps, blowers, compressors and turbines require the basic knowledge of fluid mechanics. Heating/cooling and ventilating of large underground tunnels, mines, pipeline systems, large office buildings and even private rooms require knowledge of fluid mechanics. Basic principles of the fluid mechanics are even applicable in the designs of artificial hearts, heart-ling machines, breathing aids and other such devices. All we want to do is to establish the fact that knowledge and principles of fluid mechanics are used in industry, manufacturing, and even in daily human life.

Having said this, we can now safely say that governing and studying the movement of the fluid flow in rotating or sliding cylinders is of much importance for industrial point of view.

Movement of a typical viscoelastic fluid i.e. fractional Maxwell fluid in rotating cylinder has been an area of keen interest for the theorists, researchers and mathematicians working on the fluid mechanics [1, 2]. When analyzing fluid motion, researchers are interested in use of rheological constitutive equations with fractional derivatives, in addition to the classical rheological constitutive equations. Time ordinary derivative are changed to fractional order derivative to find these equations, for fractional calculus [3]. Initially Maxwell model could not provide reasonable fit of data to be experimented for complete range of frequencies and this gave birth to Fractional Maxwell model. In comparison to the classical Maxwell model, the Fractional Maxwell fluid model can demonstrate better agreement of the data to be experimented.

Recently, among the various flow of fluids the oscillating and rotatory flows of fluid has attained significant attention. For example in [4] authors considered Exact Solutions for rotating flows of a Generalized Burgers's Fluid in Cylindrical domains where as in [5] same has been done for Oldroyd-B fluid in circular cylinder. Mainly the reason for this is that oscillating flows are more common in practical processes like towing operation, oil drilling, mixing and bioengineering. The reason for this is the flow of blood in veins due to periodic pressure gradient. In 2005 Yin and Zhu has exhibited research paper on oscillating issues of Fractional Maxwell fluid[6]. In 2012, Karam Rahaman discussed oscillating flow of upper convected Maxwell fluid and this was done by a cylinder [7]. The precise solutions for sinusoidal motion of visco- elastic non-Newtonian fluids were discovered by Mahmood *et. al* [8, 9, 10, 11, 12].

Various number of researchers are doing research since last decade to find the exact solutions of non Newtonian fluids, particulary the ones having boundary conditions of shear stress. The first ones to develop and come up with the solutions were Water *et. al* [13]. The exact solution for the Fractional Maxwell

fluid which have shear stress boundary condition and flow rotationally in circular cylinder was given by Siddique [14].
Moreover, Rajagopal came up with two very firm yet simple solutions in regard to the motion of second grade fluid which had the torsional and longitudinal oscillation of unbounded rod. Bandelli *et. al* discus unsteady motions of second grade fluid[15]. The Erdagon's work was extended by Fetecau which had association to non-Newtonian fluid to cosine and sine oscillations of the flat plate [16, 17]. The exact solution for the Maxwell fluid which had oscillatory flow in cylinder was given by Vieru. *et. al* so considering all of this Laplace and Hankel transforms are practiced in this thesis to examine torsional sine oscillations of Fractional Maxwell Fluid which endured shear stress in an infinite circular cylinder [18].
According to the best knowledge of the us no attempt has so far been made to find the solutions for the movement of fractional Maxwell fluid present within two coaxial cylinders of infinite lengths and oscillating within.
In this article we find the exact solutions for the velocity field $v(r,t)$ and shear stress $\tau(r,t)$ corresponding to the motion of a Maxwell fluid between two cylinders, which are infinite and coaxial with inner cylinder is at rest and outer one is oscillating. Laplace transform and finite Hankel transform are used to obtain their solutions. The solutions are presented under series form in term of generalized G-functions. These will satisfy governing equations and all imposed boundary and initial conditions. Furthermore the corresponding solutions for the Maxwell flow of Newtonian fluid are also obtained in term of limiting cases.

2. Mathematical Formulation and Governing Equation of Problem

The constitutive equation for the Maxwell Fluid is defined by

$$T = S - pI \ ; \quad \lambda\frac{\delta S}{\delta t} + S = \mu C_1; \tag{1}$$

S denotes the extra stress tensor and p is the pressure, I denotes unit tensor, λ is called relaxation time, μ is called dynamic viscosity and T stands for Cauchy stress tensor stands for hydrostatic pressure.

$$\frac{\delta S}{\delta t} = \frac{dS}{dt} - LS - SL^T = \dot{S} - LS - SL^T. \tag{2}$$

Assume that the Maxwell fluid is in the annulus of coaxial circular cylinders whose radii R_1 and R_2 respectively, where R_1 is less than R_2 and lengths are infinite. when $t = 0$, the fluid and two cylinders are fixed. when $t > 0$, the outer cylinder starts oscillating around their axis with velocity $W \sin(\omega\ t)$ and ω is the angular velocity of the outer cylinder. Then the velocity field $\mathbf{v}$ and the extra stress $\mathbf{S}$ is of the form

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)e_z, \tag{3}$$

$$\mathbf{S} = \mathbf{S}(r,t), \tag{4}$$

where e_z is called the unit vector in the $z-direction$.
In start, when the fluid is fixed, we have

$$\mathbf{S}(r,0)=0\ ;\qquad \mathbf{v}(r,0)=0. \tag{5}$$

The Maxwell fluid have the governing equations,

$$\Big(1+\lambda D_t^{\beta}\Big)\frac{\partial v(r,t)}{\partial t}=\nu\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\Big)v(r,t), \tag{6}$$

$$\Big(1+\lambda D_t^{\beta}\Big)\tau(r,t)=\mu\frac{\partial v(r,t)}{\partial r}. \tag{7}$$

The non-trivial shear stress is denoted by $\tau(r,t)$, λ is the material constant, μ is called the dynamic viscosity and kinematic viscosity is denoted by $\nu=\frac{\mu}{\rho}$, where ρ is constant density of the Maxwell fluid.
Furthermore, the fractional differential operator D_t^{β} of the Maxwell fluid is defined by [19]

$$D_t^{\beta}f(t)=\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t\frac{f(\tau)}{(t-\tau)^{\beta}}d\tau,\qquad 0\le\beta<1 \tag{8}$$

where $\Gamma(.)$ is the Gamma function which can be defined as

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt\qquad ;\qquad z>0 \tag{9}$$

$$v\ (r,\ 0)=0\ ;\ r\ \epsilon\ (R_1,\ R_2) \tag{10}$$

$$v\ (R_1,\ t)=0\ ,\ v\ (R_2,\ t)=V\sin(\Omega\ t)\ ,\ \text{for } t>0 \tag{11}$$

3. Calculation of the Velocity Field

Now we apply Laplace transformation to equation, we have

$$q(1+\lambda q^{\beta})\bar{v}(r,q)=\nu\bigg(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\bigg)\bar{v}(r,q), \tag{12}$$

$$\bar{v}(R_1,q)=0\qquad ,\bar{v}(R_2,q)=\frac{V\Omega}{q^2+\Omega^2}, \tag{13}$$

Here q denotes the parameter of transformation.
We can define, Hankel Transform of $\bar{v}(r,q)$ as

$$\bar{V}_H(r_n,q)=\int_{R_1}^{R_2}r\bar{v}(r,q)B_0(rr_n)dr,$$

where

$$B_0(rr_n)=J_0(rr_n)Y_0(R_2r_n)-J_0(R_2r_n)Y_0(rr_n). \tag{14}$$

Here, r_n are positive roots of $B_0(R_1r)=0$, and $J_0(.)$ and $Y_0(.)$ represent the Bessel functions whose order is zero of first and second type, respectively.
Now consider R.H.S of the equation (12), we have,

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q)B_0(rr_n)dr = \frac{2}{\pi}\frac{V\Omega}{(q^2+\Omega^2)} - r_n^2\bar{V}_H(r_n,q). \tag{15}$$

Again, from equation (12), we have the following result

$$q(1+\lambda q^\beta)\bar{V}_H(r_n,q) = \nu\left[\frac{2}{\pi}\frac{V\Omega}{q^2+\Omega^2} - r_n^2\bar{V}_H(r_n,q)\right], \tag{16}$$

Then,

$$(q+\lambda q^{\beta+1}+\nu r_n^2)\bar{V}_H(r_n,q) = \frac{2\nu}{\pi}\frac{V\Omega}{(q^2+\Omega^2)}. \tag{17}$$

Now, simplification for $\bar{V}_H(r_n,q)$

$$\bar{V}_H(r_n,q) = \frac{2}{\pi}\frac{V\Omega}{(q^2+\Omega^2)}\frac{\nu}{(q+\lambda q^{\beta+1}+\nu r_n^2)}. \tag{18}$$

we can write the above equation as,

$$\bar{V}_H(r_n,q) = \frac{2}{\pi r_n^2}\frac{V\Omega}{(q^2+\Omega^2)} - \frac{2V\Omega}{\pi r_n^2(q^2+\Omega^2)}\frac{(q+\lambda q^{\beta+1})}{(q+\lambda q^{\beta+1}+\nu r_n^2)}. \tag{19}$$

or we can say as,

$$\bar{V}_H(r_n,q) = \bar{V}_{1H}(r_n,q) - \bar{V}_{2H}(r_n,q),$$

where

$$\bar{V}_{1H}(r_n,q) = \frac{2}{\pi r_n^2}\frac{V\Omega}{(q^2+\Omega^2)},$$

and

$$\bar{V}_{2H}(r_n,q) = \frac{2V\Omega}{\pi r_n^2(q^2+\Omega^2)}\frac{(q+\lambda q^{\beta+1})}{(q+\lambda q^{\beta+1}+\nu r_n^2)}.$$

Now, going to define Inverse Hankel Transform,

$$\bar{v}(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_0^2(R_1 r_n)B_0(rr_n)}{J_0^2(R_1 r_n)-J_0^2(R_2 r_n)}\bar{V}_H(r_n,q) \tag{20}$$

Inverse Hankel Transform or $\bar{V}_{1H}(r_n,q)$ and $\bar{V}_{2H}(r_n,q)$ are

$$\bar{v}_1(r,q) = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}\frac{V\Omega}{q^2+\Omega^2}.$$

$$\bar{v}_2(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_0^2(R_1 r_n)B_0(rr_n)}{J_0^2(R_1 r_n)-J_0^2(R_2 r_n)}\bar{V}_{2H}(r_n,q).$$

Now,

$$\bar{v}(r,q)=\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\frac{V\Omega}{q^2+\Omega^2}-\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_0^2(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times\left[\frac{2V\Omega}{\pi r_n^2(q^2+\Omega^2)}\right.$$
$$\left.\frac{(q+\lambda q^{\beta+1})}{q+\lambda q^{\beta+1}+\nu r_n^2}\right]. \quad (21)$$

or,equivalently

$$\bar{v}(r,q)=\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\frac{V\Omega}{q^2+\Omega^2}-\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times\left[\frac{V\Omega}{(q^2+\Omega^2)}\right.$$
$$\left.\frac{(q+\lambda q^{\beta+1})}{q+\lambda q^{\beta+1}+\nu r_n^2}\right]. \quad (22)$$

Now by using the identity,

$$\frac{1}{q+\lambda q^{\beta+1}+\nu r_n^2}=\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{q^{-k-1}}{(q^\beta+\frac{1}{\lambda})^{k+1}}. \quad (23)$$

as,

$$\frac{1}{a+z}=\sum_{k=0}^{\infty}\frac{(-z)^k}{a^{k+1}}. \quad (24)$$

$$\bar{v}(r,q)=\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\frac{V\Omega}{q^2+\Omega^2}-\frac{\pi}{\lambda}\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\times\left[\frac{V\Omega}{(q^2+\Omega^2)}\right.$$
$$\left.\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\frac{q^{-k-1}}{(q^\beta+\frac{1}{\lambda})^{k+1}}(q+\lambda q^{\beta+1})\right], \quad (25)$$

$$\bar{v}(r,q)=\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\frac{V\Omega}{q^2+\Omega^2}-V\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\sum_{k=0}^{\infty}\left(\frac{\nu r_n^2}{\lambda}\right)^k$$
$$\times\left[\Omega\frac{q}{(q^2+\Omega^2)}\frac{q^{-k-1}}{(q^\beta+\frac{1}{\lambda})^k}\right]. \quad (26)$$

Taking inverse Laplace transformation and using the formula

$$\pounds^{-1}\left[\frac{q^b}{(q^a-d)^c}\right]=G_{a,b,c}(d,t), \quad (27)$$

and also by using the convolution theorem we get,

$$v(r,t)=\frac{\ln(r/R_1)}{\ln(R_2/R_1)}V\sin(\Omega t)-V\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k$$

$$\left[\Omega\int_0^t \cos\Omega(t-\tau)G_{\beta,-k-1,k}(-\lambda^{-1},\tau)d\tau\right]. \tag{28}$$

4. Calculation of the shear Stress

Now taking Laplace transformation of (7)
$\left[(1+\lambda D_t^{\beta})\tau(r,t)\right] = \pounds\left[\mu\frac{\partial v(r,t)}{\partial r}\right]\bar{\tau}(r,q)+\lambda q^{\beta}\bar{\tau}(r,q) = \mu\frac{\partial\bar{v}(r,q)}{\partial r}\bar{\tau}(r,q)\left[1+\lambda q^{\beta}\right] = \mu\frac{\partial\bar{v}(r,q)}{\partial r}$,

$$\bar{\tau}(r,q) = \frac{\mu}{(1+\lambda q^{\beta})}\frac{\partial\bar{v}(r,q)}{\partial r}. \tag{29}$$

we have

$$\frac{\partial\bar{v}(r,q)}{\partial r} = \frac{1}{\ln(R_2/R_1)}\frac{1}{r}\frac{V\Omega}{q^2+\Omega^2} + V\pi\sum_{n=1}^{\infty}\frac{r_nJ_0^2(R_1r_n)\tilde{B}_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}$$
$$\sum_{k=0}^{\infty}\left(\frac{\nu r_n^2}{\lambda}\right)^k\times\left[\Omega\frac{q}{(q^2+\Omega^2)}\frac{q^{-k-1}}{(q^{\beta}+\frac{1}{\lambda})^k}\right], \tag{30}$$

$$\bar{\tau}(r,q) = \frac{\mu}{(1+\lambda q^{\beta})}\left[\frac{1}{\ln(R_2/R_1)}\frac{1}{r}\frac{V\Omega}{q^2+\Omega^2} + V\pi\sum_{n=1}^{\infty}\frac{r_nJ_0^2(R_1r_n)\tilde{B}_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\right.$$
$$\left.\sum_{k=0}^{\infty}\left(\frac{\nu r_n^2}{\lambda}\right)^k\times\left(\Omega\frac{q}{(q^2+\Omega^2)}\frac{q^{-k-1}}{(q^{\beta}+\frac{1}{\lambda})^k}\right)\right], \tag{31}$$

$$\bar{\tau}(r,q) = \frac{\mu}{\lambda}\left[\frac{1}{\ln(R_2/R_1)}\frac{1}{r}\frac{V\Omega}{q^2+\Omega^2}\frac{1}{q^{\beta}+\frac{1}{\lambda}} + V\pi\sum_{n=1}^{\infty}\frac{r_nJ_0^2(R_1r_n)\tilde{B}_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\right.$$
$$\left.\sum_{k=0}^{\infty}\left(\frac{\nu r_n^2}{\lambda}\right)^k\left[\Omega\left(\frac{q}{q^2+\Omega^2}\frac{q^{-k-1}}{(q^{\beta}+\frac{1}{\lambda})^{k+1}}\right)\right]\right], \tag{32}$$

$$\tau(r,t) = \left[\frac{\mu}{\lambda}\right]\left[\frac{1}{\ln(R_2/R_1)}\frac{V}{r}\int_0^t \sin\Omega(t-\tau)G_{\beta,0,1}(-\lambda^{-1},\tau)d\tau+\right.$$
$$V\pi\sum_{n=1}^{\infty}\frac{r_nJ_0^2(R_1r_n)\tilde{B}_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}$$
$$\left.\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\times\left(\Omega\int_0^t \cos\Omega(t-\tau)G_{\beta,-k-1,k+1}(-\lambda^{-1},\tau)d\tau\right)\right]. \tag{33}$$

5. Limiting Cases

5.1. Ordinary Maxwell Fluid. By putting $\alpha \to 1$,$\beta \to 1$ in the above results, **Velocity Field**:

$$v(r,t) = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}(V\sin\Omega t) - V\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times\left[\Omega\int_0^t \cos\Omega(t-\tau)G_{1,-k-1,k}(-\lambda^{-1},\tau)d\tau\right]. \quad (34)$$

Shear Stress:

$$\tau(r,t) = \left[\frac{\mu}{\lambda}\right]\left[\frac{1}{\ln(R_2/R_1)}\frac{V}{r}\int_0^t \sin\Omega(t-\tau)G_{1,0,1}(-\lambda^{-1},\tau)d\tau + V\pi\sum_{n=1}^{\infty}\frac{r_nJ_0^2(R_1r_n)\tilde{B}_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)} \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times\left(\Omega\int_0^t \cos\Omega(t-\tau)G_{1,-k-1,k+1}(-\lambda^{-1},\tau)d\tau\right)\right]. \quad (35)$$

5.2. Newtonian Fluid. By placing $\lambda \to 0$, $\lambda_r \to 0$ in the last consequences of ordinary Maxwell fluid, we can get the results for newtonian fluid.

$$v(r,t) = \frac{\ln(r/R_1)}{\ln(R_2/R_1)}(V\sin\Omega t) - V\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)} \left(\frac{-\Omega\nu r_n^2}{\Omega^2+\nu^2r_n^4}\exp(-\nu r_n^2t) + \frac{\Omega\nu r_n^2}{\Omega^2+\nu^2r_n^4}\cos\Omega t + \frac{\Omega^2}{\Omega^2+\nu^2r_n^4}\sin\Omega t\right). \quad (36)$$

Above result is similar to the result which already established in [1].

$$\tau(r,t) = \mu\left[\frac{1}{r}\frac{1}{\ln(R_2/R_1)}(V\sin\Omega t) + V\pi\sum_{n=1}^{\infty}\frac{r_nJ_0^2(R_1r_n)\tilde{B}_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)} \left(\frac{-\Omega\nu r_n^2}{\Omega^2+\nu^2r_n^4}\exp(-\nu r_n^2t) + \frac{\Omega\nu r_n^2}{\Omega^2+\nu^2r_n^4}\cos\Omega t + \frac{\Omega^2}{\Omega^2+\nu^2r_n^4}\sin\Omega t\right)\right]. \quad (37)$$

6. Conclusion

The purpose of this work is to find exact solutions for velocity field and adequate shear stress corresponding to the flow of Maxwell fluid between two longitudinally oscillating circular cylinders, whose lengths are infinite, with fractional derivatives. The motion of the fluid is produced by outer cylinder where inner cylinder is fixed, at time $t = 0^+$. It is worthwile to note that results obtained in [1] are special cases of our results. Finally we recover corresponding solutions for ordinary Maxwell and Newtonian fluids are as limiting cases.

7. Appendix

Followings are some expressions used in the text:
(A1). The finite Hankel transform of the funcion

$$a(r) = \frac{C_1 \ln(R_2/r) + C_2 \ln(r/R_1)}{\ln(R_2/R_1)} \; .$$

satisfying $a(R_1) = C_1$ and $a(R_2) = C_2$ is

$$a_n(r) = \int_{R_1}^{R_2} r a(r) B_0(r r_n) dr = \frac{2C_2}{\pi r_n^2} - \frac{2C_1}{\pi r_n^2} \frac{J_0(R_2 r_n)}{J_0(R_1 r_n)}.$$

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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