Journal of Scientific Research & Reports
5(4): 285-294, 2015; Article no.JSRR.2015.096
ISSN: 2320-0227

SCIENCEDOMAIN *international*
www.sciencedomain.org

Analytical Solutions of Strongly Non-linear Problems by the Iteration Perturbation Method

A. M. El-Naggar[1] and G. M. Ismail[2*]

[1]Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt.
[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag, 82524, Egypt.

Authors' contributions

This work was carried out in collaboration between both authors. Author GM designed the study, wrote the protocol, wrote the first draft of the manuscript and managed the experimental process. Author AM managed the literature searches; analyses of the study performed the compression analysis. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JSRR/2015/14364
Editor(s):
(1) Narayan Thapa, Department of Mathematics and CS, Minot State University, USA.
Reviewers:
(1) Anonymous, Pakistan.
(2) Anonymous, Bangladesh.
(3) Anonymous, Pakistan.
(4) Anonymous, Greece.
Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=750&id=22&aid=7521

Original Research Article

Received 28th September 2014
Accepted 5th December 2014
Published 27th December 2014

ABSTRACT

This paper applies modified He's iteration perturbation method to study periodic solutions of strongly nonlinear oscillators. Some examples are given to illustrate the effectiveness and convenience of the method. The results are compared with the numerical solution and the comparison showed a proper accuracy of this method.

Keywords: He's iteration perturbation method; periodic solutions; strongly non-linear oscillators.

1. INTRODUCTION

Obviously the study of nonlinear systems and their behavior remains one of the most important aspects of engineering, applied mathematics, physics and other scientific fields. These nonlinear systems are real physical systems which are modeled by nonlinear differential equations, for this reason, one cannot overemphasize the importance of understanding

**Corresponding author: Email: gamalm2010@yahoo.com;*

and gaining useful insights towards the behavior of these nonlinear differential equations so as to make accurate and precise decisions while working with real physical systems.

On the other hand, nonlinear differential equations are very difficult and complex to study. Due to their complexity, it is very difficult and most times impossible to obtain an exact solution to these nonlinear differential equations. Over the years, researchers have developed many tools that will aid in the study of these nonlinear differential equations. Methods like perturbation methods [1,2,3], numerical methods [4], and most recently the approximate methods have been developed in order to understand the behavior of these nonlinear differential equations. Many approximate methods have evolved recently, among them are, Iteration method [5,6], Frequency-amplitude formulation [7-9], Energy balance method [10-12], Variational iteration method [13,14], Homotopy perturbation method [15-19] Parameter expanding method [20-22] and Hamiltonian approach [23,24].

With the rapid development of nonlinear science, it appears an ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems. Though it is easy for us now to find solutions to linear systems by means of numerical simulations, it is still very difficult to solve nonlinear problems analytically. The study of nonlinear oscillators has been important in the development of the theory of dynamical systems. The Van der Pol oscillator can be regarded as describing a mass-spring-damper system with a nonlinear position-dependent damping coefficient or, equivalently, an RLC electrical circuit with a negative nonlinear resistor, and has been used to develop models in many applications, such as electronics, biology or acoustics. It represents a nonlinear system with an interesting behavior that arises naturally in several applications. Also Duffing oscillators are described by nonlinear differential equations that modeled the behavior of many practical problems that arise in engineering, physics, and in many real world applications [1,2,13,25]. It is well-known that Duffing oscillators can be found in the modeling of free vibrations of a restrained uniform beam with intermediate lumped mass, the nonlinear dynamics of slender elastica, the generalized Pochhammer-Chree (PC) equation, the generalized compound KdV equation in nonlinear wave systems, among others [26]. Duffing's equation is a model of many structural systems, it is regarded as one of the most important differential equations because it appears in various physical and engineering problems such that, nonlinear optics and plasma physics.

In this paper only the first order approximations is considered as it gives results with enough technical accuracy.

2. THE ITERATION METHODS

Consider the following nonlinear equation

$$\ddot{x} + \varepsilon f(x, \dot{x}, \ddot{x}, t) = 0, \tag{1}$$

Where ε is a constant parameter and $f(x, \dot{x}, \ddot{x}, t)$ is nonlinear analytical function.
We introduce the variable $y = dx/dt$, and then Eq. (1) can be replaced by an equivalent system

$$\dot{x}(t) = y(t), \qquad \dot{y}(t) = -\varepsilon f(x, y, \dot{y}, t). \tag{2}$$

Assume that its initial approximate guess can be expressed as

$$x(t) = A\cos(\omega t) \tag{3}$$

Where ω is the angular frequency of oscillation. Then we have

$$\dot{x}(t) = -A\omega\sin(\omega t) = y(t), \qquad \ddot{x}(t) = -A\omega^2\cos(\omega t) = \dot{y}(t). \tag{4}$$

Substituting Eqs. (3) and (4) into Eq. (2), we obtain

$$\dot{y}(t) = -\varepsilon f(x, y, \dot{y}, t) = -\varepsilon\left[\sum_{n=0}^{\infty}\alpha_{2n+1}\sin(2n+1)\omega t + \sum_{m=0}^{\infty}\beta_{2m+1}\cos(2m+1)\omega t\right]. \tag{5}$$

Substituting Eq. (5) into Eq. (2)2, yields

$$\dot{y}(t) = -\varepsilon\left[\alpha_1\sin\omega t + \alpha_3\sin 3\omega t + ... + \beta_1\cos\omega t + \beta_3\cos 3\omega t + ...\right]. \tag{6}$$

Integrating Eq. (6), gives

$$y(t) = -\frac{\varepsilon}{\omega}\left[-\alpha_1\cos\omega t - \frac{1}{3}\alpha_3\cos 3\omega t - ... + \beta_1\sin\omega t + \frac{1}{3}\beta_3\sin 3\omega t + ...\right]. \tag{7}$$

Comparing Eqs. (4)1 and (7), we obtain the angular frequency ω and eliminating the secular term from Eq. (7) we can obtain the amplitude A of the oscillation, after them integrating Eq. (7) yields the first order approximate solution of Eq. (1)

$$x(t) = \frac{\varepsilon}{\omega^2}\left[\alpha_1\sin\omega t + \frac{1}{9}\alpha_3\sin 3\omega t + ... + \beta_1\cos\omega t + \frac{1}{9}\beta_3\cos 3\omega t + ...\right]. \tag{8}$$

The objective of this method is to eliminate dependent variable in succession until there remains only a single equation containing only one dependent variable. After the solution of remaining equation has been found, the other dependent variables can be found in turn, using the original differential equation or those that have appeared in the elimination process.

3. APPLICATIONS

In order to assess advantages and the accuracy of the iteration method, we should consider the following examples.

3.1 Example 1

As a first application, let us consider a classical Van der Pol oscillator with a nonlinear damping function of higher polynomial order, which models many physical problems, likes, an electrical circuit with a triode valve, and was later extensively studied as a host of a rich class of dynamical behavior, including relaxation oscillations, quasi periodicity, elementary bifurcations, and chaos [27]. The model considered is a classical Van der Pol oscillator with a nonlinear damping function of higher polynomial order described by the following nonlinear equation [28].

$$\ddot{x} + x - \varepsilon(1 - x^2 + x^4 - x^6)\dot{x} = 0. \tag{9}$$

Eq. (9) can be separated to the two differential equations of the first order

$$\dot{x} = y, \tag{10}$$

$$\dot{y} = -x + \varepsilon(1 - x^2 + x^4 - x^6)y. \tag{11}$$

Supposing that the exact solution of Eq. (9) is

$$x = A\cos\omega t, \text{ then } \dot{x} = y = -A\omega\sin\omega t, \tag{12}$$

Where A is the amplitude of the limit cycle and ω is the nonlinear frequency, substituting x and y into the right hand side of Eq. (11), we have

$$\left.\begin{aligned}\dot{y} &= -A\cos\omega t + \varepsilon A\omega\left(\tfrac{5A^6}{64} - \tfrac{A^4}{8} + \tfrac{A^2}{4} - 1\right)\sin\omega t + \tfrac{\varepsilon A^3\omega}{4}\left(\tfrac{9A^4}{16} - \tfrac{3A^2}{4} + 1\right)\sin 3\omega t \\ &\quad - \tfrac{\varepsilon A^5\omega}{16}\left(1 - \tfrac{5A^2}{4}\right)\sin 5\omega t + \tfrac{\varepsilon A^7\omega}{64}\sin 7\omega t.\end{aligned}\right\} \tag{13}$$

Integrating (13) yields

$$\left.\begin{aligned}y &= -\tfrac{A}{\omega}\sin\omega t - \varepsilon A\left(\tfrac{5A^6}{64} - \tfrac{A^4}{8} + \tfrac{A^2}{4} - 1\right)\cos\omega t - \tfrac{\varepsilon A^3}{12}\left(\tfrac{9A^4}{16} - \tfrac{3A^2}{4} + 1\right)\cos 3\omega t \\ &\quad + \tfrac{\varepsilon A^5}{80}\left(1 - \tfrac{5A^2}{4}\right)\cos 5\omega t - \tfrac{\varepsilon A^7}{448}\cos 7\omega t.\end{aligned}\right\} \tag{14}$$

Comparing (12) and (14) we get $\omega = 1$. Eq. (14) may be written as

$$\left.\begin{aligned}y &= -A\sin t - \varepsilon A\left(\tfrac{5A^6}{64} - \tfrac{A^4}{8} + \tfrac{A^2}{4} - 1\right)\cos t - \tfrac{\varepsilon A^3}{12}\left(\tfrac{9A^4}{16} - \tfrac{3A^2}{4} + 1\right)\cos 3t \\ &\quad + \tfrac{\varepsilon A^5}{80}\left(1 - \tfrac{5A^2}{4}\right)\cos 5t - \tfrac{\varepsilon A^7}{448}\cos 7t.\end{aligned}\right\} \tag{15}$$

Avoiding the presence of a secular term in Eq. (15) needs

$$\frac{5A^6}{64} - \frac{A^4}{8} + \frac{A^2}{4} - 1 = 0, \tag{16}$$

then we obtain the amplitude

$$A = 1.56183, \tag{17}$$

which leads the same value as illustrated in Yamapi *et al*. [28] by using Lindstedt's perturbation method (LPM). Thus Eq. (15) reduces to

$$y = -A\sin t - \frac{\varepsilon A^3}{12}\left(\frac{9A^4}{16} - \frac{3A^2}{4} + 1\right)\cos 3t + \frac{\varepsilon A^5}{80}\left(1 - \frac{5A^2}{4}\right)\cos 5t - \frac{\varepsilon A^7}{448}\cos 7t. \tag{18}$$

Integrating (18) yields the first order approximate solution of Eq. (9)

$$x = A\cos t - \frac{\varepsilon}{9}\left(\frac{A^3}{4} - \frac{3A^5}{16} + \frac{9A^7}{64}\right)\sin 3t + \frac{\varepsilon}{25}\left(\frac{A^5}{16} - \frac{5A^7}{64}\right)\sin 5t - \frac{\varepsilon A^7}{3136}\sin 7t. \tag{19}$$

Eq. (9) is numerically solved and the result is compared with analytic one (19). In Fig. 1 the both solutions are plotted. Comparing the solutions it is evident that they are in a good agreement.

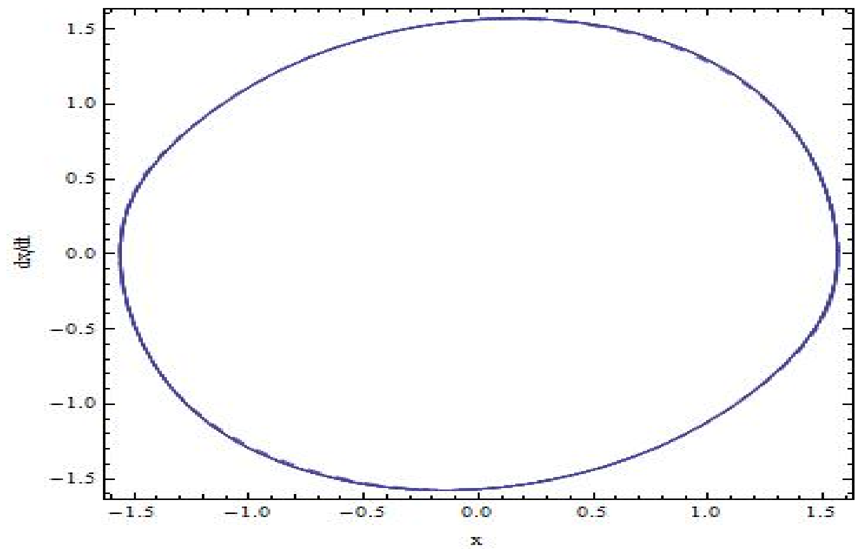

Fig. 1. comparison of the approximate solution (——) with the numerical solution (- - -) for $\varepsilon = 0.1$

3.2 Example 2

The model considered is the Rayleigh oscillator with excitation term in the form

$$\ddot{x} + x + \varepsilon(1 - \dot{x}^2)\dot{x} = \varepsilon\, \dot{x}^2 \sin t. \tag{20}$$

We re-write Eq. (20) in the form

$$\dot{x} = y, \tag{21}$$

$$\dot{y} = -x - \varepsilon(1 - y^2)y + \varepsilon y^2 \sin t, \tag{22}$$

supposing that

$$x = A\cos\omega t, \text{ then } \quad y = -A\omega \sin\omega t. \tag{23}$$

Substituting from Eq. (23) into the right hand side of Eq. (22), we have

$$\left.\begin{aligned} \dot{y} = & -A\cos\omega t + \varepsilon A\omega\left(1 - \tfrac{3A^2\omega^2}{4}\right)\sin\omega t + \tfrac{\varepsilon A^2\omega^2}{2}\sin t + \tfrac{\varepsilon A^3\omega^3}{4}\sin 3\omega t \\ & + \tfrac{\varepsilon A^2\omega^2}{4}\sin(2\omega - 1)t - \tfrac{\varepsilon A^2\omega^2}{4}\sin(2\omega + 1)t. \end{aligned}\right\} \tag{24}$$

Integrating (24) yields

$$\left.\begin{aligned} y = & -\tfrac{A}{\omega}\sin\omega t - \varepsilon A\left(1 - \tfrac{3A^2\omega^2}{4}\right)\cos\omega t - \tfrac{\varepsilon A^2\omega^2}{2}\cos t - \tfrac{\varepsilon A^3\omega^2}{12}\cos 3\omega t \\ & - \tfrac{\varepsilon A^2\omega}{4(2\omega - 1)}\cos(2\omega - 1)t + \tfrac{\varepsilon A^2\omega}{4(2\omega + 1)}\cos(2\omega + 1)t. \end{aligned}\right\} \tag{25}$$

Comparing (23) and (25), we get $\omega = 1$. We now re-write Eq. (25) as follows

$$y = -A\sin t - \varepsilon A\left(1 + \frac{3A}{4} - \frac{3A^2}{4}\right)\cos t + \frac{\varepsilon A^2}{12}(1 - A)\cos 3t. \tag{26}$$

No secular term requires that,

$$\frac{3A^2}{4} - \frac{3A}{4} - 1 = 0. \tag{27}$$

Then we obtain the amplitude

$$A = 1.75831, \tag{28}$$

thus Eq. (26) reduces to

$$y = -A\sin t + \frac{\varepsilon A^2}{12}(1-A)\cos 3t. \tag{29}$$

Integrating Eq. (29) yields the first order approximate solution of Eq. (20)

$$x = A\cos t + \frac{\varepsilon A^2}{36}(1-A)\sin 3t. \tag{30}$$

The analytical result (30) and result obtained by numerical integration of Eq. (20) are compared. The two solutions are in a good agreement as shown in Fig. 2.

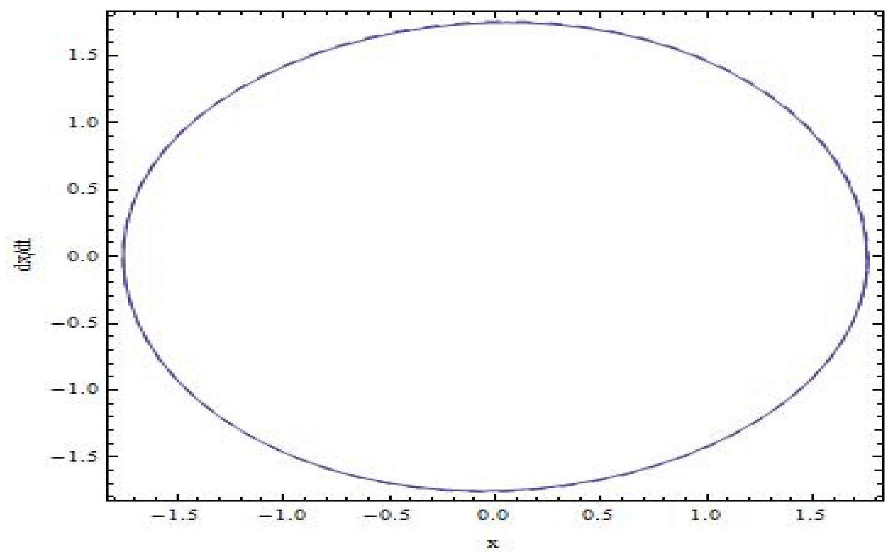

Fig. 2. Comparison of the approximate solution (——) with the numerical solution (- - -) for $\varepsilon = 0.1$

3.3 Example 3

We consider the strongly nonlinear oscillator

$$\ddot{x} + x + \varepsilon(1 - x^2 - \dot{x}^2)\dot{x} = \varepsilon x\dot{x}\cos t. \tag{31}$$

Eq. (31) is equivalent to the two non-autonomous systems

$$\dot{x} = y, \tag{32}$$

$$\dot{y} = -x - \varepsilon(1 - x^2 - y^2 - x\cos t)y, \tag{33}$$

supposing that

$$x = A\cos\omega t, \text{ then } \quad y = -A\omega\sin\omega t. \tag{34}$$

Substituting from Eq. (34) into the right hand side of Eq. (33), we have

$$\left.\begin{aligned}\dot{y} &= -A\cos\omega t + \varepsilon A\omega\left(1 - \tfrac{A^2}{4} - \tfrac{3A^2\omega^2}{4}\right)\sin\omega t + \tfrac{\varepsilon A^3\omega}{4}(\omega^2 - 1)\sin 3\omega t \\ &\quad - \tfrac{\varepsilon A^2\omega}{4}\sin(2\omega+1)t - \tfrac{\varepsilon A^2\omega}{4}\sin(2\omega-1)t.\end{aligned}\right\} \tag{35}$$

Integrating Eq. (35) yields

$$\left.\begin{aligned} y &= -\tfrac{A}{\omega}\sin\omega t - \varepsilon A(1-\tfrac{A^2}{4}-\tfrac{3A^2\omega^2}{4})\cos\omega t - \tfrac{\varepsilon A^3}{12}(\omega^2-1)\cos 3\omega t \\ &+ \tfrac{\varepsilon A^2}{4(2\omega+1)}\cos(2\omega+1)t + \tfrac{\varepsilon A^2}{4(2\omega-1)}\cos(2\omega-1)t. \end{aligned}\right\} \tag{36}$$

Comparing (33) and (34) yields $\omega = 1$.

We now re-write Eq. (36) as follows

$$y = -A\sin t - \varepsilon A(1-\frac{A}{4}-A^2)\cos t + \frac{\varepsilon A^2}{12}\cos 3t. \tag{37}$$

Avoiding the secular term in Eq. (37) needs

$$A^2 + \frac{A}{4} - 1 = 0, \tag{38}$$

then we obtain the amplitude

$$A = 0.882782, \tag{39}$$

thus Eq. (37) reduces to

$$y = -A\sin t + \frac{\varepsilon A^2}{12}\cos 3t. \tag{40}$$

Integrating (40) yields the first order approximate solution of Eq. (31)

$$x = A\cos t + \frac{\varepsilon A^2}{36}\sin 3t. \tag{41}$$

Comparing result (41) with that obtained by numerical integration of Eq. (31). In Fig. 3 it is shown that the two solutions are in a good agreement.

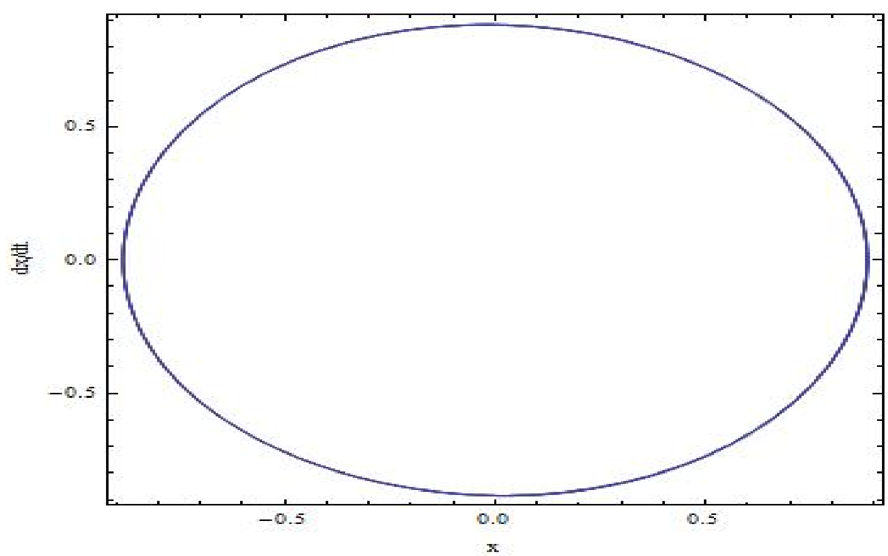

Fig. 3. Comparison of the approximate solution (——) with the numerical solution (- - -) for $\varepsilon = 0.1$

3.4 Example 4

Now we will consider the following Duffing oscillator with 5th-order nonlinearity studied by [29,30]:

$$\ddot{x} + x + \varepsilon x^5 = 0. \tag{42}$$

Eq. (42) can be separated to the two differential equations of the first order

$$\dot{x} = y, \tag{43}$$

$$\dot{y} = -x - \varepsilon x^5. \tag{44}$$

Supposing that the exact solution of Eq. (42) is

$$x = A\cos\omega t, \text{ then } \dot{x} = y = -A\omega\sin\omega t, \tag{45}$$

Where A is the amplitude of the limit cycle and ω is the nonlinear frequency, substituting x from Eq. (45) into the right hand side of Eq. (44), we have

$$\dot{y} = -A\left(1+\frac{5}{8}\varepsilon A^4\right)\cos\omega t - \frac{5\varepsilon A^5}{16}\cos 3\omega t - \frac{\varepsilon A^5}{16}\cos 5\omega t. \tag{46}$$

Integrating (46) yields

$$y = -\frac{A}{\omega}\left(1+\frac{5}{8}\varepsilon A^4\right)\sin\omega t - \frac{5\varepsilon A^5}{48\omega}\sin 3\omega t - \frac{\varepsilon A^5}{80\omega}\sin 5\omega t. \tag{47}$$

Comparing (45) and (47) we get $\omega = \sqrt{1+\frac{5}{8}\varepsilon A^4}$ Which is the same as given by (Max-min frequency see Ibsen *et al*. [29] and parameter-expanding method see Zengin *et al*. [30]). Integrating (47) yields the first order approximate solution of Eq. (42)

$$x = \frac{A}{\omega^2}\left(1+\frac{5}{8}\varepsilon A^4\right)\cos\omega t + \frac{5\varepsilon A^5}{144\omega^2}\cos 3\omega t + \frac{\varepsilon A^5}{400\omega^2}\cos 5\omega t. \tag{48}$$

The analytical result (48) and result obtained by numerical integration of Eq. (42) are compared. The two solutions are in a good agreement as shown in Fig. 4.

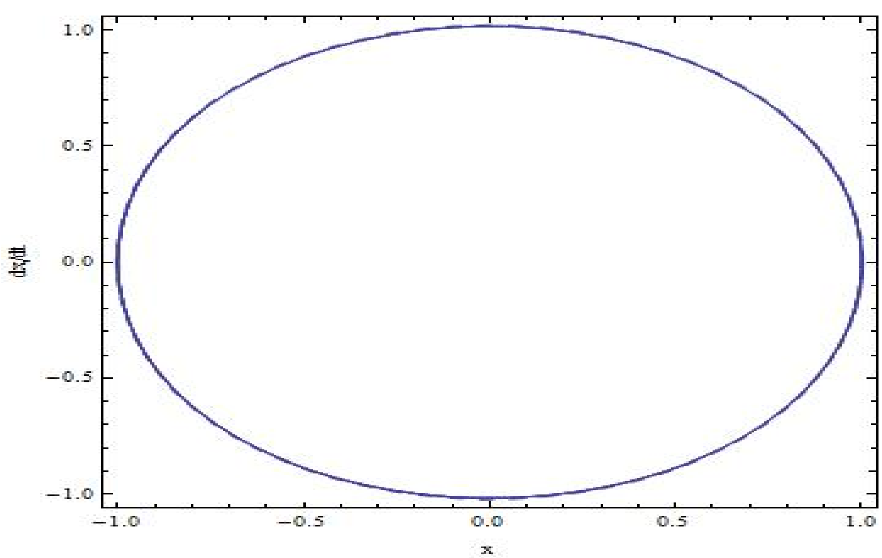

Fig. 4. Comparison of the approximate solution (——) with the numerical solution (- - -) for $\varepsilon = 0.1$

4. CONCLUSION

In this paper, the modified iteration perturbation method has been implemented in order to analyze the equation of nonlinear oscillators; we have shown the effectiveness and efficiency of the iteration perturbation method in obtaining analytic approximate solutions to nonlinear differential equations. All the examples show that the presented results are in excellent agreement with those obtained by the numerical results obtained by using Runge-Kutta method. The general conclusion is that the iteration perturbation method provides an easy and direct procedure for determining approximations of periodic solutions to nonlinear oscillators.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Nayfeh AH. Problems in perturbation. Wiley New York; 1985.
2. Nayfeh AH. Perturbation methods. Wiley New York; 2000.
3. Hinch EJ. Perturbation methods. Cambridge Texts in Applied Mathematics, Cambridge University Press Cambridge; 1991.
4. Boyce WE. Diprima RC. Elementary differential equations and boundary value problems. John Wiley & Sons, Inc; 2000.
5. He JH. Limit cycle and bifurcation of nonlinear problems. Chaos Solitons & Fractals. 2005;26:827-833.
6. Özis T, Yldrm A, Generating the periodic solutions for forcing van der Pol oscillators by the Iteration Perturbation method. Nonlinear Analysis: Real World Applications. 2009;10:1984-1989.
7. He JH. Comment on He's frequency formulation for nonlinear oscillators. European Journal of Physics. 2008;29:19-22.
8. He JH. An improved amplitude-frequency formulation for nonlinear oscillators, International. Journal of Nonlinear Science and Numerical Simulation. 2008;9:211-212.
9. Davodi AG, Gangi DD, Azami R, Babazadeh H. Application of improved amplitude-frequency formulation to nonlinear differential equation of motion equations. Modern Physics Letters B. 2009;23:3427-3436.
10. Mehdipour I, Ganji DD, Mozaffari M. Application of the energy balance method to nonlinear vibrating equations. Current Applied Physics. 2010;10:104-112.
11. Younesian D, Askari H, Saadatnia Z, Kalami Yazdi M. Frequency analysis of strongly nonlinear generalized duffing oscillators using He's frequency-amplitude formulation and He's energy balance method. Computers and Mathematics with Applications. 2010;59:3222-3228.
12. Askari H, Kalami Yazdi M, Saadatnia Z. Frequency analysis of nonlinear oscillators with rational restoring force via He's energy balance method and He's variational approach. Nonlinear Science Letters A. 2010;1:425-430.
13. He JH. Variational iteration method-a kind of nonlinear analytical technique: some examples. Int. J. Non-Linear. Mech. 1999; 34:699-708.
14. Herisanu N, Marinca V. A modified variational iteration method for strongly nonlinear problems. Nonlinear Science Letters A. 2010;1:183-192.
15. He JH. New interpretation of homotopy perturbation method. International Journal of Modern Physics B. 2006;20:2561-2568.
16. He JH. A coupling method of a homotopy technique and a perturbation technique for non-linear problems. International Journal of Non-Linear Mechanics. 2000;35:37-43.
17. He JH. Application of homotopy perturbation method to nonlinear wave equations. Chaos Solitons and Fractals. 2005;26:695-700.
18. Ganji DD. The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer. Physics Letters A. 2006;355:337-341.
19. Cveticanin L, Abd El-Latif GM, El-Naggar AM, Ismail GM. Periodic solutions of the generalized Rayleigh equation. J. Sound. Vibration. 2008;818:580-591.
20. He JH. Some asymptotic methods for strongly nonlinear equations. International Journal of Modern Physics B. 2006;20:1141-1199.
21. Xu L. He's parameter-expanding methods for strongly nonlinear oscillators. Journal of Computational and Applied Mathematics. 2007;207:148-154.
22. Xu L. Determination of limit cycle by He's

parameter-expanding methods for strongly nonlinear oscillators. Journal of Sound and Vibration. 2007;302:178-184.
23. Khan NA, Riaz F, Khan NA. Parameters approach applied on nonlinear oscillators, shock and vibration. Shock and Vibration. 2014;8. Article ID 624147,
24. Khan NA, Khan NA, Riaz F. Dynamic analysis of rotating pendulum by hamiltonian approach. Chinese Journal of Mathematics. 2013;4. Article ID 237370.
25. Arnold MD. An efficient solution for scattering by a perfectly conducting strip grating. J. Electromagnetic Waves and Applications. 2006;20:891-900.
26. Wu BS, Sun WP, Lim CW. An analytical approximate technique for a class of strongly non-linear oscillators. International Journal of Non-Linear Mechanics. 2006; 41:766-774.
27. Atay FM. Van der Pol's oscillator under delayed feedback. J. Sound Vibr. 1998;218:333-339.
28. Yamapi R, Nana BR, Enjieu HG. Dynamics and active control of motion of a driven multi limit cycle Van der Pol oscillators, Int. J. Bifurcation. Chaos. 2007;17:1343-1354.
29. Ibsen LB, Barari A, Kimiaeifar A. Analysis of highly nonlinear oscillation systems using He's max-min method and comparison with homotopy analysis and energy balance methods. Sādhanā. 2010;35:433-448.
30. Zengin FO, Kaya MO, Demirbag SA. Approximate period calculation for some strongly nonlinear oscillation by He's parameter-expanding methods. Nonlinear Analysis: Real World Applications. 2009;10:2177-2182.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history.php?iid=750&id=22&aid=7521