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Compacton, Peakon and Solitary Wave Solutions of the Osmosis $K(3,2)$ Equation

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Authors' contributions

This work was carried out in collaboration between all authors. Authors Jing Chen and JC designed the study, worked out the expressions of the compacton and peakon and wrote the first draft of the manuscript. Author JZ performed the simulations of the solitary wave and compacton solutions, and presented the figures of all the solutions by Maple. Author JC and Jing Chen revised the manuscript according the referees' review reports. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, the bifurcation method of planar systems and simulation method of differential equations are employed to investigate the bounded traveling wave solutions of the osmosis $K(3,2)$ equation. Our results show that this equation admits a variety of physical solutions such as compacton, peakon and smooth solitary wave solutions.

Keywords: $K(3,2)$ *equation; compacton; peakon; solitary wave solution; bifurcation method.*

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1. INTRODUCTION

In recent years, many nonlinear partial differential equations (NLPDEs) have been derived from physics, mechanics, engineering, biology, chemistry and other fields. Since exact solutions can help people know deeply the described process and possible applications, seeking exact solutions is of great importance for NLPDEs.

In [1], the authors investigated the role of nonlinear dispersion in the formation of patterns in liquid drops by introducing and studying a special type of KdV equation, of the form

$$u_t + \left(u^m\right)_x + \left(u^n\right)_{xxx} = 0, m > 1, 1 \le n \le 3, \quad (1.1)$$

Which are now named the $K(m,n)$. They introduced a class of solitary waves with compact support, which they called compactons, that collide elastically and vanish identically outside a finite core region. Many powerful methods were applied to seek the exact or numerical solutions of Eq. (1.1), such as finite element method [2], finite difference method [3], Adomain method [4], homotopy perturbation method [5], Exp-function method [6], variational iteration method [7], bifurcation method [8] and G'/G method [9].

In [10], the authors presented the application of Adomian decomposition method to the nonlinear osmosis $K(m,n)$ equations:

$$u_t + \left(u^m\right)_x - \left(u^n\right)_{xxx} = 0, \qquad m, n > 1. \quad (1.2)$$

They chose two special cases, $K(2,2)$ and $K(3,3)$ equations to illustrate the scheme such that new exact solutions with solitary patterns are of important significance and developed the new exact solutions which are generated by combining two distinct solutions of the $K(2,2)$ and $K(3,3)$ equations. At last, they established the general formulas for exact solutions of equations $K(m,n)$ when $m = n$ being even and odd integers for $n > 1$. The peakon, soliton, and other new types of traveling wave solutions to the osmosis $K(2,2)$ equation was obtained in [15, 16,17].

The purpose of this paper is intend to investigate the bounded traveling-wave solutions to the Osmosis $K(3,2)$ equation

$$u_t + \left(u^3\right)_x - \left(u^2\right)_{xxx} = 0. \quad (1.3)$$

We'll apply the bifurcation method of planar systems to obtained the compacton, peakon and solitary wave solutions. Note that this equation was not discussed in [10,11,12,13]. To the best of our knowledge, the traveling wave solutions to this equation have not been obtained yet. In [8] the authors obtain the peakons and smooth periodic wave solutions for the original $K(3,2)$ equation

$$u_t + \left(u^3\right)_x + \left(u^2\right)_{xxx} = 0. \quad (1.4)$$

In comparison, our work in this paper not only obtain the peakon solution, but also obtain the compacton solution to the osmosis $K(3,2)$ equation. In addition, we derive the existence and properties of solitary wave solutions, and make the numerical simulations of the solitary wave and compacton by Maple, which can verify the correction of our theoretical analysis. Note also that in [14], the authors investigate the traveling-wave solutions of the BBM-like $B(3,2)$ equation

$$u_t + \alpha u_x + \beta\left(u^3\right)_x - \gamma\left(u^2\right)_{xxt} = 0. \quad (1.5)$$

They obtained the elliptic periodic blow-up solutions trigonometric periodic blow-up solutions, symmetric elliptic periodic wave solutions, hyperbolic smooth solitary wave solution, hyperbolic blow-up solutions, hyperbolic peakon wave solution and hyperbolic periodic peakon wave solutions. One can see that, under the traveling-wave transformation $u = \varphi(x-ct)$ $= \varphi(\xi)$, the obtained traveling-wave system are similar. However, there have some differences between ours and theirs. Firstly, the osmosis $K(3,2)$ equation we studied is of different physical background from the $B(3,2)$ equation. Our results show that the osmosis $K(3,2)$ equation has 'dark' solitary wave, compacton, and peakon, while the work in [14] shows that the

$B(3,2)$ equation has 'bright' solitary wave and peakon. Secondly, we obtain a compacton solution (such solution is a typical solitary-wave solution for the well-known $K(m,n)$ equation, see [1]) for the osmosis $K(3,2)$ equation, while they didn't give this solution to the $B(3,2)$ equation. Thirdly, we make the numerical simulations of the solitary wave solution and compacton solution, which verify the correction of our theoretical analysis, while they didn't.

The remainder of the paper is organized as follows. In Section 2, based on an independent variable transformation [8], we investigate the $K(3,2)$ equation by the bifurcation method of planar systems. In Section 3, applying the qualitative theory of polynomial differential system we obtain the existence and properties of solitary wave solutions theoretically. Numerical simulation of the solitary wave solution is made by Maple. In Section 4, in certain region of the parametric space, exact compacton solution is obtained and numerical simulation of such solution is also made by Maple. In Section 5, expression of exact peakon in certain parametric region is derived. A short conclusion is given in Section 6.

2. BIFURCATION AND PHASE PORTRAITS OF TRAVELING WAVE SYSTEM

Let $u=\varphi(\xi)+\mu$ with $\xi=x-ct$ be the solution of Eq. (1.3), then it follows that

$$-c\varphi'+\left((\varphi+u)^3\right)'-\left((\varphi+u)^2\right)'''=0, \tag{2.1}$$

Where c represents the wave speed and μ interprets physically the level of the undisturbed wave surface at infinity.

Integrating (2.1) once we have

$$\varphi^3+3\mu\varphi^2+3\left(\mu^2-c\right)\varphi-2\left(\varphi'\right)^2-2(\varphi+\mu)\varphi''+\mu^3=g, \tag{2.2}$$

Where g is a constant of integration.

Taking $g=\mu^3$ in (2.2) we rewrite it as the planar autonomous system

$$\begin{cases}\dfrac{d\varphi}{d\xi}=y\\[2mm] \dfrac{dy}{d\xi}=\dfrac{\varphi^3+3\mu\varphi^2+3\left(\mu^2-c\right)\varphi-2y^2}{2(\varphi+\mu)}\end{cases}, \tag{2.3}$$

Note that (2.3) has a singular line $\varphi=-\mu$ .To avoid the line temporarily we make transformation $d\xi=(\varphi+\mu)d\varsigma$. Under this transformation, Eq. (2.3) becomes

$$\begin{cases}\dfrac{d\varphi}{d\varsigma}=2(\varphi+\mu)y,\\[2mm] \dfrac{dy}{d\varsigma}=\varphi^3+3\mu\varphi^2+3\left(\mu^2-c\right)\varphi-2y^2,\end{cases} \tag{2.4}$$

Which possesses a Hamiltonian

$$H(\varphi,\gamma)=\varphi^2\left[\frac{1}{5}\varphi^3+\mu\varphi^2+\frac{1}{3}(6\mu^2-c)\varphi+\frac{\mu}{2}(3\mu^2-c)\right]-(\varphi+\mu)^2\gamma^2=h, \tag{2.5}$$

Where h is a constant. System (2.4) has the same topological phase portraits as system (2.3) except for the straight line $\varphi=-\mu$ .Obviously, $\varphi=-\mu$ is a constant solution of (2.2) with $g=\mu^3$. For a fixed h , (2.5) determines a set of invariant curves of system (2.4). As h is varied, (2.5) determines different families of orbits of system (2.4) having different dynamical behaviors. Let $M(\varphi_e,y_e)$ be the coefficient matrix of the linearized system of (2.4) at the equilibrium point (φ_e,y_e), then

$$M(\varphi_e,y_e)=\begin{pmatrix}2y_e & 2(\varphi_e+\mu)\\ 3\varphi^2+6\mu\varphi+\left(3\mu^2-c\right) & -4y_e\end{pmatrix} \tag{2.6}$$

and at this equilibrium point, we have

$$J(\varphi_e,\gamma_e)=\det M(\varphi_e,\gamma_e)=-8\gamma_e^2-2(\varphi_e+\mu)[3\varphi^2+6\mu\varphi+(3\mu^2-c)] \tag{2.7}$$

$$p(\varphi_e,y_e)=trace\left(M(\varphi_e,y_e)\right)=-2y_e \tag{2.8}$$

By the theory of planar dynamical system (cf.

[15,16]), for an equilibrium point of a planar dynamical system, if $J > 0$, then this equilibrium point is a saddle point; it is a center point if $J > 0$ and $p = 0$; I $J = 0$ and the Poincar´e index of the equilibrium point is 0, then it is a cusp. Then we obtain the bifurcation curves as follows:

$$c = \frac{3}{4}\mu^2, c = \frac{9}{32}\left(13 - \sqrt{105}\right)\mu^2 \; (\mu < 0), c = \mu^2, c = \frac{9}{5}\mu^2 \; (\mu < 0),$$

$$c = 3\mu^2, c = \frac{9}{32}\left(13 + \sqrt{105}\right)\mu^2 \; (\mu < 0), \mu = 0,$$

Which divide the (μ, c)-parameter plane into 22 subregions:

$$B_1 = \left\{(\mu, c) : \mu < 0, c < \frac{3}{4}\mu^2\right\}, \; B_2 = \left\{(\mu, c) : \mu < 0, c = \frac{3}{4}\mu^2\right\},$$

$$B_3 = \left\{(\mu, c) : \mu < 0, \frac{3}{4}\mu^2 < c < \frac{9}{32}\left(13 - \sqrt{105}\right)\mu^2\right\}, \; B_4 = \left\{(\mu, c) : \mu < 0, c = \frac{9}{32}\left(13 - \sqrt{105}\right)\mu^2\right\},$$

$$B_5 = \left\{(\mu, c) : \mu < 0, \frac{9}{32}\left(13 - \sqrt{105}\right)\mu^2 < c < \mu^2\right\}, \; B_6 = \left\{(\mu, c) : \mu < 0, c = \mu^2\right\},$$

$$B_7 = \left\{(\mu, c) : \mu < 0, \mu^2 < c < 3\mu^2\right\}, \; B_8 = \left\{(\mu, c) : \mu < 0, c = 3\mu^2\right\},$$

$$B_9 = \left\{(\mu, c) : \mu < 0, c > 3\mu^2\right\}, \; B_{10} = \left\{(\mu, c) : \mu > 0, c < \frac{3}{4}\mu^2\right\},$$

$$B_{11} = \left\{(\mu, c) : \mu < 0, c = \frac{3}{4}\mu^2,\right\}, \; B_{12} = \left\{(\mu, c) : \mu > 0, \frac{3}{4}\mu^2 < c < \mu^2\right\},$$

$$B_{13} = \left\{(\mu, c) : \mu > 0, c = \mu^2,\right\}, \; B_{14} = \left\{(\mu, c) : \mu > 0, \mu^2 < c < \frac{9}{5}\mu^2\right\},$$

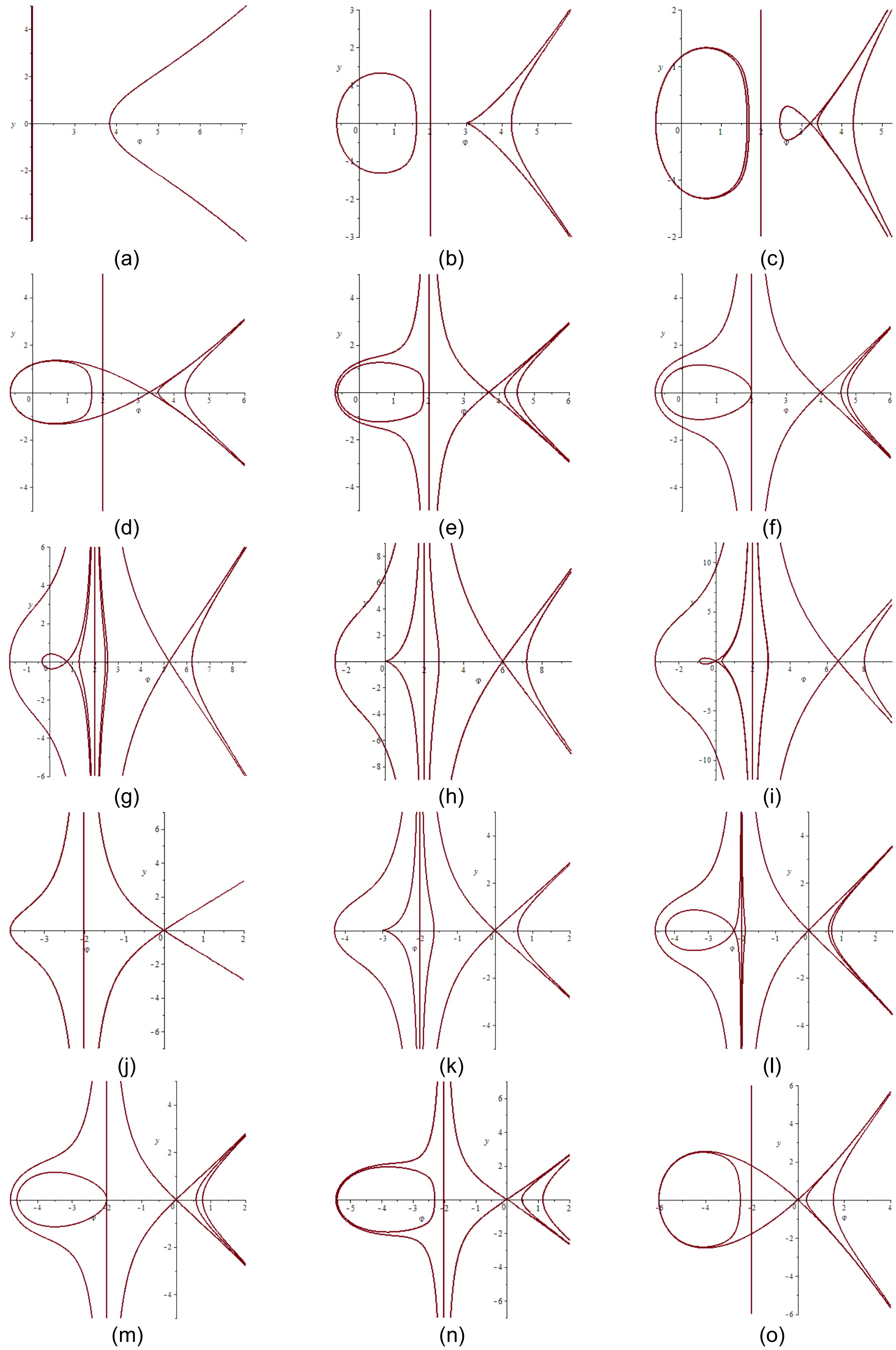
$$B_{15} = \left\{(\mu, c) : \mu > 0, c = \frac{9}{5}\mu^2,\right\}, \; B_{16} = \left\{(\mu, c) : \mu > 0, \frac{9}{5}\mu^2 < c < 3\mu^2\right\},$$

$$B_{17} = \left\{(\mu, c) : \mu > 0, c = 3\mu^2,\right\}, \; B_{18} = \left\{(\mu, c) : \mu > 0, 3\mu^2 < c < \frac{9}{32}\left(13 + \sqrt{105}\right)\mu^2\right\},$$

$$B_{19} = \left\{(\mu, c) : \mu > 0, c = \frac{9}{32}\left(13 + \sqrt{105}\right)\mu^2,\right\}, \; B_{20} = \left\{(\mu, c) : \mu > 0, c > \frac{9}{32}\left(13 + \sqrt{105}\right)\mu^2\right\},$$

$$B_{21} = \left\{(\mu, c) : \mu = 0, c > 0,\right\}, \; B_{22} = \left\{(\mu, c) : \mu = 0, c < 0\right\}.$$

We present the bifurcation sets and phase portraits of system (2.3) in Fig. 1.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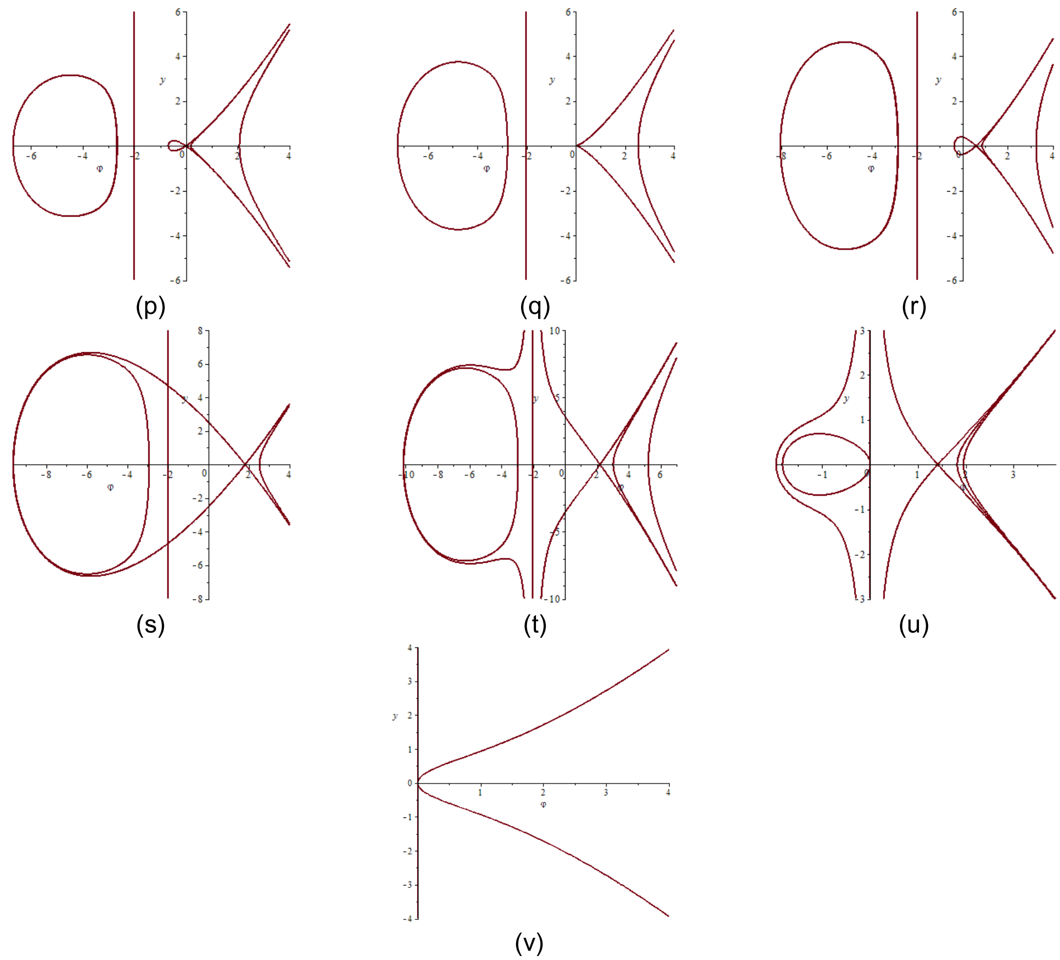

Fig. 1. The bifurcation sets and phase portraits of system (2.3). (a) $(\mu,c)\in B_1$ **. (b)** $(\mu,c)\in B_2$ **. (c)** $(\mu,c)\in B_3$ **. (d)** $(\mu,c)\in B_4$ **. (e)** $(\mu,c)\in B_5$ **. (f)** $(\mu,c)\in B_6$ **. (g)** $(\mu,c)\in B_7$ **. (h)** $(\mu,c)\in B_8$ **. (i)** $(\mu,c)\in B_9$ **. (j)** $(\mu,c)\in B_{10}$ **. (k)** $(\mu,c)\in B_{11}$ **. (l)** $(\mu,c)\in B_{12}$ **. (m)** $(\mu,c)\in B_{13}$ **. (n)** $(\mu,c)\in B_{14}$ **. (0)** $(\mu,c)\in B_{15}$ **. (p)** $(\mu,c)\in B_{16}$ **. (q)** $(\mu,c)\in B_{17}$ **. (r)** $(\mu,c)\in B_{18}$ **. (s)** $(\mu,c)\in B_{19}$ **. (t)** $(\mu,c)\in B_{20}$ **. (u)** $(\mu,c)\in B_{21}$ **. (v)** $(\mu,c)\in B_{22}$

3. SOLITARY WAVE SOLUTION

Using the theory of planar dynamical system (cf. [15,16]) and the Hartman-Grobman Theorem [17], we obtain the following existence result and basic properties for solitary wave solutions of (1.3):

Proposition 3.1. If $(\mu,c)\in B_3$ falls into the parametric subregions: $B_3, B_7, B_9, B_{12}, B_{16}, B_{18},$, then Eq.(1.3) has solitary wave solutions of depression, which are obtained from the homoclinic connection based at the saddle point of (2.3). These solutions are symmetric with respect to the trough. The solitary waves tend exponentially to a constant on either side of the trough.

When $(\mu,c)\in B_3$ (that is, μ, c satisfy $\mu<0$, $\frac{3}{4}\mu^2<c<\frac{9}{32}\left(13-\sqrt{105}\right)\mu^2$), (2.3) has a homoclinic orbit, which can be expressed as:

$$\varphi^2\left[\frac{1}{5}\varphi^3+\mu\varphi^2+\frac{1}{3}(6\mu^2-c)\varphi+\frac{\mu}{2}(3\mu^2-c)\right]-(\varphi+\mu)^2y^2=H(\varphi_1,0), \quad (3.1)$$

Where $(\varphi_1,0)$ is a saddle point of (2.3) and $\varphi_1=\frac{-3\mu+\sqrt{4c-3\mu^2}}{2}$. Regarding y in (3.1) as a function of φ, yields an explicit expression of the homoclinic connection in the phase plane. Generally, substituting this expression into the first equation in (2.3) and integrating along the homoclinic orbit, we can obtain the expression of the solitary wave solution. However, since the explicit expression of the homoclinic orbit is much complicated, we cannot obtain the expression of the solitary wave solution explicitly.

Now we take a set of data and employ mathematical software Maple to numerically simulate the solitary wave solution. Taking $\mu=-2.0, c=3.05$, then the homoclinic orbit leaves the critical saddle point $(\varphi_1,0)=(3.223606798,0)$ and crosses the horizontal axis once at $(2.468173816,0)$, before returning to the saddle point symmetrically with respect to the horizontal axis. Thus we take the initial conditions $\varphi(0)=2.468173816$ and $\varphi'(0)=0$ and use Maple to simulate the integral curve of Eq. (2.2) as in Fig. 2. We can see that our theoretic analysis agrees with the numerical simulation.

When (μ,c) falls into the parametric subregions: B_7, B_9, B_{12}, B_{16} and B_{18}, respectively, we can analyze in the same manner. We omit the details for simplicity.

4. COMPACTON SOLUTION

Due to the fact that the planar system (2.3) is discontinuous along the straight line $\varphi=-\mu$ (for a detailed account on various evolution equations arising in the context of nonlinear water waves which yield so-called singular nonlinear traveling wave systems we refer to [18]), (1.3) has the compactly supported solitary waves. The orbit connecting the equilibrium point and tangent to the invariant $\varphi=-\mu$, corresponds to the compactly supported solitary wave (compacton) and has finite existence time, cf. Fig. 1(f), (m),(u).

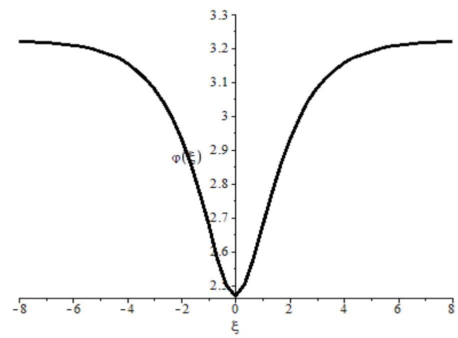

Fig. 2. Simulation of the solitary wave solution of (1.3) with $\mu=-2.0$, $c=3.05$, $\varphi(0)=2.468173816$ and $\varphi'(0)=0$

When $(\mu,c)\in B_6$ (that is, satisfy $\mu<0$, $c=\mu^2$), (2.3) has a closed orbit connecting the equilibrium point $(-\mu,0)$, which can be expressed as:

$$y=\pm\frac{1}{\sqrt{15}}\sqrt{(\varphi+\mu)(3\varphi^2+6\mu\varphi-2\mu^2)}, \quad \varphi_2\le\varphi\le-\mu, \quad (4.1)$$

where $\varphi_2=\left(-1+\frac{\sqrt{15}}{3}\right)\mu$. Substituting (4.1) into the first equation in (2.3) and integrating along the orbit, we obtain the expression of the compacton solution

$$\xi=\pm\sqrt{15}\left(F(\varphi)-F(\varphi_2)\right), \quad \varphi_2\le\varphi\le-\mu, \quad (4.2)$$

Where

$$F(\varphi)=\pm\frac{2\times15^{-\frac{1}{4}}}{\sqrt{\mu}}EllipticF\left(\left(\frac{3}{5}\right)^{\frac{1}{4}}\sqrt{-\frac{\varphi+\mu}{\mu}},I\right), \quad (4.3)$$

$I^2=-1$ and Elliptic F is the incomplete elliptic integral of the first kind. The orbit has finite existence time, which is given by

$$T=\sqrt{15}\left(F(-\mu)-F(\varphi_2)\right). \quad (4.4)$$

We may extend this solution to the real line by setting $\varphi(\xi)=-\mu$ for $\xi\in R/(-T,T)$. This is possible since $\varphi=-\mu$ is a constant solution of

(2.2). A typical of such solution is shown in Fig. 3.

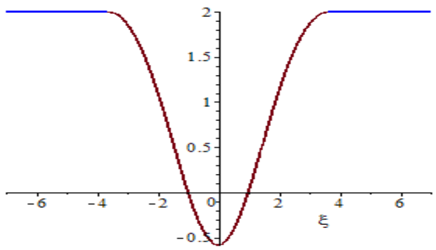

Fig. 3. Compacton solution with $\mu=-2.0, c=4.0$

Taking $\mu=-2.0, c=4.0$, then the orbit leaves the equilibrium point $(-\mu,0)=(4.0,0)$ and crosses the horizontal axis once at $(\varphi_2,0)=(-0.5819888975,0)$ before returning to the equilibrium point symmetrically with respect to the horizontal axis. The time it takes the orbit to get from $(-0.5819888975,0)$ to the equilibrium point $(-\mu,0)=(4.0,0)$ is given by $T=3.648771985$. Thus we take the initial conditions $\varphi(0)=-0.5819888975$ and $\varphi'(0)=0$ and use Maple to simulate the integral curve of Eq. (2.2) as in Fig. 4. Comparing Fig. 4 with Fig. 3, we can also see that our theoretic result agrees with the numerical simulation.

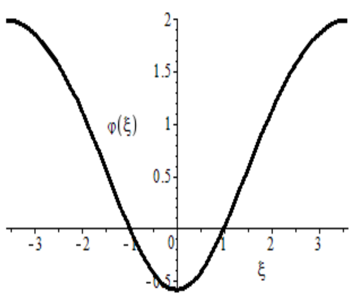

Fig. 4. Simulation of the compacton solution with $\mu=-2.0, c=4.0$,

$\varphi(0)=-0.5819888975$ **and** $\varphi'(0)=0$

When (μ,c) falls into the parametric subregions: B_{13} and B_{21}, respectively, one can obtained the similar results. We omit the detailed results for brevity.

5. PEAKON SOLUTION

From Fig.1 (d), (o) and (s), one can see that the two heteroclinic orbits connecting with saddle point and the two saddle points on singular line $\varphi=-\mu$, together with the finite line between these two saddle points corespond to the peakon solution.

When $(\mu,c)\in B_{15}$ (that is, satisfy $\mu>0$, $c=\frac{9}{5}\mu^2$), there are two heteroclinic orbits connecting with saddle point $(0,0)$ and the two saddle points $\left(-\mu,\sqrt{\frac{2}{5}\mu^3}\right)$ and $\left(-\mu,-\sqrt{\frac{2}{5}\mu^3}\right)$ on singular line $\varphi=-\mu$. Their expressions are

$$y=\pm\frac{1}{\sqrt{5}}\varphi\sqrt{\varphi+3\mu},\quad -\mu\le\varphi\le 0 \tag{5.1}$$

Substituting (5.1) into the first equation in (2.3) and integrating along the orbit, we obtain the expression of the peakon solution

$$\xi=\left|-\frac{2}{\sqrt{15\mu}}\left(\operatorname{arctan}h\left(\sqrt{\frac{\varphi+3\mu}{3\mu}}\right)-\operatorname{arctan}h\left(\sqrt{\frac{2}{3}}\right)\right)\right|,\quad -\mu\le\varphi\le 0 \tag{5.2}$$

By use of $u(x,t)=\varphi(\xi)+\mu$, we can obtain the peakon solution of Eq.(1.3). The profile of (5.2) is shown in Fig. 5.

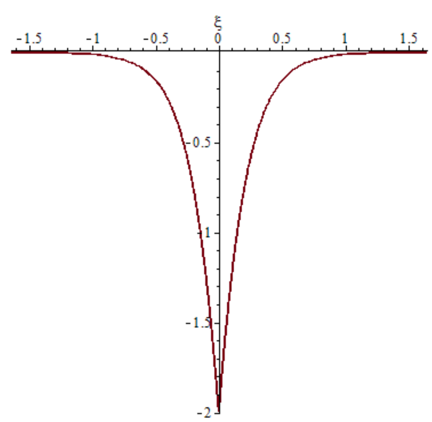

Fig. 5. The peakon solution with $\mu=2.0$ **and** $c=7.2$

When (μ, c) falls into the parametric sub regions: B_4 and B_{19}, respectively, the results are similar and we omit them here.

6. CONCLUSION

In this paper, with the aim of bifurcation method of planar systems and simulation method of differential equations are employed to investigate the compacton, peakon and solitary wave solutions of the osmosis $K(3,2)$ equation. The existence and properties of solitary wave solution, and expression of compacton in certain parametric region are obtained and numerical simulations of these solutions are made. Our results show that the theoretical analysis agrees with the simulations. The expression of exact peakon in certain parametric region is also derived. To the best of our knowledge, the obtained solutions are new for the osmosis $K(3,2)$ equation.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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