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Cantor's Ternary Set Formula-Basic Approach

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Abstract

Georg Cantor (1845-1918) introduced the notion of the cantor set, which consists of points along a single line segment with a number of remarkable and deep properties. This paper aims to emphasize a proceeding to obtain the Cantor (ternary) set, C by means of the so-called elimination of the open-middle third at each step using a general basic approach in constructing the set.

Keywords: Open middle-third; interval; cantor set; geometry;point-set topology.

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1 Introduction

The Cantor Set, named after Georg Cantor (1845-1918) for his discovery of the set has since its introduction, had a lot of applications in many branches of Mathematics, above all in Set theory, Fractal theory and Chaotic dynamical systems [1]. Cantor introduced the set in a footnote to a statement saying that perfect sets do not need to be everywhere dense [2]. His introduction of this set attracted lots of criticisms and reject from his teachers and staunch followers [3]. But today, thanks to the work of several people we have come to understand, appreciate and laud his work for its ingenuity. Although there are still obstacles to overcome pertaining to some of the deep

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and unique properties of this set, the current surge in interest for research about this set gives us reason to be optimistic. One intriguing behaviour of this Cantor Set worth mentioning here is with the devil's staircase [4],[5]. It is interesting to note however that, the consideration of this set together with others helped lay the foundation of what we now have as, point-set topology [6]. As much as possible, the text would be introduced in a form that might even be accessible to a non-specialist. In the process of doing that, we shall touch on some preliminaries to give the reader a better appreciation of the content before we employ a basic general approach for the construction among many others, after which a profound conjecture would be made.

1.1 Preliminaries

Among the many different definitions relating to the Cantor set, below would be some simple definitions we would stick to, to aid our understanding:

Definition 1.1. The **Cantor Set** is a set of points lying on a single line segment with a number of remarkable and deep properties [6].

Definition 1.2. The set obtained by repeatedly deleting the open-middle third from the closed interval [0, 1] is termed as the **Cantor Ternary Set** [6].

This is usually written as $C = \bigcap\{C_i \mid i \in N\}$ where C_i are suitable closed subsets of C_o=[0,1].Interestingly, definitions 1 and 2 may be used interchangeably as they are synonymous.

Definition 1.3. A subset of $\mathbb{R}$ of the form A=$[a_1, a_2] = \{x \mid a_1 \leqslant x \leqslant a_2$, where $a_1, a_2 \in \mathbb{R}\}$ is called a **closed real interval** [7].

Definition 1.4. Given real numbers a and b for which $a < b$,we define $(a, b) = \{x \mid a < x < b\}$, and say a point in (a, b) lies between a and b. We call a nonempty set **I** of real numbers an **interval** provided for any two points in **I**,all the points that lie between these points also belong to **I** p.9, [8].

Definition 1.5. We define a **topological space** as follows:

Let X be a set, then a **topology**, T on X is a collection of subsets of X each called an **open** set such that the following conditions hold:

(i) X and the empty set, ϕ are open;
(ii) the intersection of finitely many open sets is open;
(iii)the union of any arbitrary collection of open sets is open,
The set X together with a topology T on X is called a **topological space** [9].

Definition 1.6. A subset A of a topological space X is **closed** iff the set $X - A$ is open [9].

Definition 1.7. A subset A of a topological space X is called **dense** iff $Cl(A)$=X [9].

Definition 1.8. The **closure of A**,denoted by $Cl(A)$ is the intersection of all closed sets containing A [9].

Definition 1.9. Let (X, T) be a topological space. A set A $\subset X$ is said to be **nowhere dense** iff the interior of the closure of A is void [9].

Definition 1.10. A topological space X is said to be **compact** if every open cover of X has a finite subcover [9].

Definition 1.11. A **perfect set** is a closed set with no isolated points [10].

Definition 1.12. A topological space, X is **connected**, if it cannot be broken down into distinct pieces that form the union [9].

Definition 1.13. A subset $A \subseteq X$ of a topological space,X is said to be **disconnected** if it is not connected [11].

Definition 1.14. A set E is said to be **finite** provided either it is empty or there is a natural number,n for which E is equipotent to $\{1, ..., n\}$. We say that E is **countably infinite** provided E is equipotent to the set $\mathbb{N}$ of natural numbers. A set that is either **finite or countably infinite** is said to be **countable**. A set that is not **countable** is called **uncountable** [8].

We would denote a set function of a collection of sets by **m**, to be called **Lebesgue measurable sets** which has the following 3 properties

(i) **The measure of an interval is its length** Each nonempty interval I is Lebesgue measurable and $m(I) = l(I)$

(ii) **Measure is translation invariant** If E is Lebesgue measurable and y is any number, then the translate of E by y, $E + y = \{x + y \mid x \in E\}$, also is Lebesgue measurable and $m(E + y) = m(E)$

(iii) **Measure is countably additive over countable disjoint unions of set** If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets then,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k) \quad [8].$$

It is however worthy to note that, the Cantor set breeds paradoxes and that makes its properties deep and interesting.

1.2 Some Properties of the Cantor Set

1. The cantor set has no interval
2. The cantor set is non-empty
3. The cantor set is closed and nowhere dense
4. The cantor set is compact
5. The cantor set is perfect and totally disconnected
6. The cantor set is uncountable [12]

2 Results and Discussion

We begin with the closed real interval C_o, where the nth set formula is given as:

$$C_n = \frac{C_{n-1}}{3} \bigcup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$$

Where $n = 1, 2, 3, \dots$ and $C_o = [0, 1] \implies$ When $n = 1$

$$C_1 = \frac{C_0}{3} \bigcup \left(\frac{2}{3} + \frac{C_0}{3}\right) \tag{2.1}$$

$$C_1 = \left(\frac{0}{3}, \frac{1}{3}\right) \bigcup \left(\frac{2}{3} + \left[\frac{0}{3}, \frac{1}{3}\right]\right) \tag{2.2}$$

Open middle third $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$

$$C_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right]$$

$\implies$ When n=2

$$C_2 = \frac{C_1}{3} \bigcup \left(\frac{2}{3} + \frac{C_1}{3}\right) \tag{2.3}$$

$$C_2 = \frac{[0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]}{3} \bigcup \left(\frac{2}{3} + \frac{[0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]}{3}\right) \tag{2.4}$$

Open middle third $I_2 = (\frac{1}{9}, \frac{2}{9}) \bigcup (\frac{7}{9}, \frac{8}{9})$

$$C_2 = \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right]$$

$\implies$ When $n = 3$

$$C_3 = \frac{C_2}{3} \bigcup \left(\frac{2}{3} + \frac{C_2}{3}\right) \tag{2.5}$$

$$C_3 = \frac{[0, \frac{1}{9}] \bigcup [\frac{2}{9}, \frac{1}{3}] \bigcup [\frac{2}{3}, \frac{7}{9}] \bigcup [\frac{8}{9}, 1]}{3} \bigcup \left(\frac{2}{3} + \frac{[0, \frac{1}{9}] \bigcup [\frac{2}{9}, \frac{1}{3}] \bigcup [\frac{2}{3}, \frac{7}{9}] \bigcup [\frac{8}{9}, 1]}{3}\right) \tag{2.6}$$

Open middle third $I_3 = (\frac{1}{27}, \frac{2}{27}) \bigcup (\frac{7}{27}, \frac{8}{27}) \bigcup (\frac{19}{27}, \frac{20}{27}) \bigcup (\frac{25}{27}, \frac{26}{27})$

$$C_3 = \left[0, \frac{1}{27}\right] \bigcup \left[\frac{2}{27}, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{7}{27}\right] \bigcup \left[\frac{8}{27}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{19}{27}\right] \bigcup \left[\frac{20}{27}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, \frac{25}{27}\right] \bigcup \left[\frac{26}{27}, 1\right]$$

$\implies$ When n=4

$$C_4 = \frac{C_3}{3} \bigcup \left(\frac{2}{3} + \frac{C_3}{3}\right)$$

$$C_4 = \frac{[0, \frac{1}{27}] \bigcup [\frac{2}{27}, \frac{1}{9}] \bigcup [\frac{2}{9}, \frac{7}{27}] \bigcup [\frac{8}{27}, \frac{1}{3}] \bigcup [\frac{2}{3}, \frac{19}{27}] \bigcup [\frac{20}{27}, \frac{7}{9}] \bigcup [\frac{8}{9}, \frac{25}{27}] \bigcup [\frac{26}{27}, 1]}{3} \bigcup$$

$$\left(\frac{2}{3} + \frac{[0, \frac{1}{27}] \bigcup [\frac{2}{27}, \frac{1}{9}] \bigcup [\frac{2}{9}, \frac{7}{27}] \bigcup [\frac{8}{27}, \frac{1}{3}] \bigcup [\frac{2}{3}, \frac{19}{27}] \bigcup [\frac{20}{27}, \frac{7}{9}] \bigcup [\frac{8}{9}, \frac{25}{27}] \bigcup [\frac{26}{27}, 1]}{3}\right)$$

Much concentration is needed as it is becoming more and more cumbersome already. Open middle third $I_4 = (\frac{1}{81}, \frac{2}{81}) \bigcup (\frac{7}{81}, \frac{8}{81}) \bigcup (\frac{19}{81}, \frac{20}{81}) \bigcup (\frac{25}{81}, \frac{26}{81}) \bigcup (\frac{61}{81}, \frac{62}{81}) \bigcup (\frac{73}{81}, \frac{74}{81}) \bigcup (\frac{79}{81}, \frac{80}{81})$

$$C_4 = \left[0, \frac{1}{81}\right] \bigcup \left[\frac{2}{81}, \frac{1}{27}\right] \bigcup \left[\frac{2}{27}, \frac{7}{81}\right] \bigcup \left[\frac{8}{81}, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{19}{81}\right] \bigcup \left[\frac{20}{81}, \frac{7}{27}\right] \bigcup \left[\frac{8}{27}, \frac{25}{81}\right] \bigcup \left[\frac{26}{81}, \frac{1}{3}\right] \bigcup$$
$$\left[\frac{2}{3}, \frac{55}{81}\right] \bigcup \left[\frac{56}{81}, \frac{19}{27}\right] \bigcup \left[\frac{20}{27}, \frac{61}{81}\right] \bigcup \left[\frac{62}{81}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, \frac{73}{81}\right] \bigcup \left[\frac{74}{81}, \frac{25}{27}\right] \bigcup \left[\frac{26}{27}, \frac{79}{81}\right] \bigcup \left[\frac{80}{81}, 1\right]$$

Remark:As n increases,per the formula it gets more and more cumbersome and care must be taken to note the technicalities that arise as seen throughout.The danger may be that one may leave out a particular middle third which would in effect affect the answer,

Let's see when n=5

$$C_5 = \frac{C_4}{3} \bigcup \left(\frac{2}{3} + \frac{C_4}{3}\right)$$

We write out C_4 from the above and substitute into C_5, Open middle third I_5 is:

$$\left(\tfrac{1}{243},\tfrac{2}{243}\right)\cup\left(\tfrac{7}{243},\tfrac{8}{243}\right)\cup\left(\tfrac{19}{243},\tfrac{20}{243}\right)\cup\left(\tfrac{25}{243},\tfrac{26}{243}\right)\cup\left(\tfrac{61}{243},\tfrac{62}{243}\right)\cup\left(\tfrac{73}{243},\tfrac{74}{243}\right)\cup\left(\tfrac{79}{243},\tfrac{80}{243}\right)\cup\left(\tfrac{163}{243},\tfrac{164}{243}\right)\cup$$

$$\left(\tfrac{169}{243},\tfrac{170}{243}\right)\cup\left(\tfrac{181}{243},\tfrac{182}{243}\right)\cup\left(\tfrac{187}{243},\tfrac{188}{243}\right)\cup\left(\tfrac{217}{243},\tfrac{218}{243}\right)\cup\left(\tfrac{223}{243},\tfrac{224}{243}\right)\cup\left(\tfrac{235}{243},\tfrac{236}{243}\right)\cup\left(\tfrac{241}{243},\tfrac{242}{243}\right)$$

$$C_5=\left[0,\tfrac{1}{243}\right]\cup\left[\tfrac{2}{243},\tfrac{1}{81}\right]\cup\left[\tfrac{2}{81},\tfrac{7}{243}\right]\cup\left[\tfrac{8}{243},\tfrac{1}{27}\right]\cup\left[\tfrac{2}{27},\tfrac{19}{243}\right]\cup\left[\tfrac{20}{243},\tfrac{7}{81}\right]\cup\left[\tfrac{8}{81},\tfrac{25}{243}\right]\cup\left[\tfrac{26}{243},\tfrac{1}{9}\right]\cup\left[\tfrac{2}{9},\tfrac{55}{243}\right]\cup$$

$$\left[\tfrac{56}{243},\tfrac{19}{81}\right]\cup\left[\tfrac{20}{81},\tfrac{61}{243}\right]\cup\left[\tfrac{62}{243},\tfrac{7}{27}\right]\cup\left[\tfrac{8}{27},\tfrac{73}{243}\right]\cup\left[\tfrac{74}{243},\tfrac{25}{81}\right]\cup\left[\tfrac{26}{81},\tfrac{79}{243}\right]\cup\left[\tfrac{80}{243},\tfrac{1}{3}\right]\cup\left[\tfrac{2}{3},\tfrac{163}{243}\right]\cup\left[\tfrac{164}{243},\tfrac{55}{81}\right]\cup$$

$$\left[\tfrac{56}{81},\tfrac{169}{243}\right]\cup\left[\tfrac{170}{243},\tfrac{19}{27}\right]\cup\left[\tfrac{20}{27},\tfrac{181}{243}\right]\cup\left[\tfrac{182}{243},\tfrac{61}{81}\right]\cup\left[\tfrac{62}{81},\tfrac{187}{243}\right]\cup\left[\tfrac{188}{243},\tfrac{7}{9}\right]\cup\left[\tfrac{8}{9},\tfrac{217}{243}\right]\cup\left[\tfrac{218}{243},\tfrac{73}{81}\right]\cup$$

$$\left[\tfrac{74}{81},\tfrac{223}{243}\right]\cup\left[\tfrac{224}{243},\tfrac{25}{27}\right]\cup\left[\tfrac{26}{27},\tfrac{235}{243}\right]\cup\left[\tfrac{236}{243},\tfrac{79}{81}\right]\cup\left[\tfrac{80}{81},\tfrac{241}{243}\right]\cup\left[\tfrac{242}{243},1\right]$$

After a careful discussion, we suggested a **conjecture** which could be probed further

2.1 Conjecture(Translation invariance preserves properties of the Cantor set)

The Cantor set is Lebesgue measurable and hence invariant under translation with all its properties preserved.

3 Conclusion

Per the conjecture above, what goes wrong if only measurability properties are maintained for the Cantor Set but its other properties are not? This is an area which should be further looked into. Also, from the above construction, we can say that the general basic approach can be used to construct the ternary set. This approach can however be modified to reduce:

- computational effort and
- computational time

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Competing Interests

The authors declare that no competing interests exist.

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